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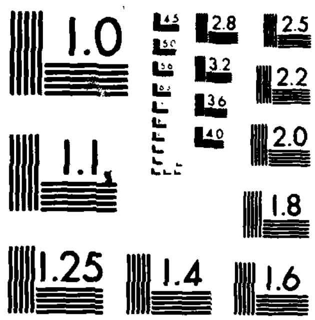

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INITIAL-BOUNDARY VALUE PROBLEMS FOR LINEAR HYPERBOLIC SYSTEMS

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ABSTRACT

We discuss and interpret a theory developed by Kreiss and others for studying the suitability of boundary conditions for linear hyperbolic systems of partial differential equations. The existing theory is extremely technical.

The present discussion is based on the characteristic variety of the system. The concept of characteristic variety leads to

(1) a physical interpretation of the theory in terms of wave propagation, and

(2) a physical and geometrical method for visualizing the algebraic structure of the system. The great complexity of the theory is caused by certain aspects of this structure.

We also point out connections between the above work and a corresponding theory regarding the stability of finite difference approximations.

AMS (MOS) Subject Classifications: 35L50

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SIGNIFICANCE AND EXPLANATION

This paper is concerned with boundary conditions for linear hyperbolic systems of partial differential equations in several space dimensions. For various reasons, it can be difficult to determine whether a given boundary condition is suitable for a given system. The existing theory which deals with this issue is quite complicated and algebraic. In the present paper we describe how this theory can be interpreted in terms of concepts associated with wave propagation. We also mention connections with an analogous theory regarding the stability of finite difference approximations.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

INITIAL-BOUNDARY VALUE PROBLEMS FOR LINEAR HYPERBOLIC SYSTEMS

Robert L. Higdon*

1. Introduction.

In this paper we discuss boundary conditions for linear hyperbolic partial differential equations in several space dimensions. Our goal is to provide some interpretations and explanations of a theory developed by Kreiss and others (e.g., [10],[14],[15],[20],[21]) for determining whether a given set of conditions is suitable for a given equation. The theory is of fundamental importance. However, it is extremely complicated, and its physical interpretation is not immediately apparent. In the present section we describe the context of the theory and outline the interpretations which we will give regarding it. At the end of the section we point out some relationships between this theory and the study of finite difference approximations.

Examples of hyperbolic equations include the Euler equations of gas dynamics, the shallow water equations, Maxwell's equations, equations of magnetohydrodynamics, and the classical wave equation. Except for the wave equation, these examples are systems of first-order equations. The first three cases will be discussed in Section 4.

The theory to be discussed here deals with linear first-order systems. Equations of higher order can be reduced to first-order systems by standard techniques (e.g., John [9], Taylor [26]). Of the above examples, Maxwell's equations and the wave equation are linear. The theory is applicable to linearized versions of the others.

The unknown, dependent variables in the problems of interest are functions of space and time, e.g., $u = u(x,t)$ where $x \in R^n$. The spatial variable x is typically confined to a subdomain of R^n. For example, a fluid may flow in a region which is bounded by a solid wall. In other problems the spatial domain of interest may be bounded, at least in part, by an open, artificial boundary. These boundaries are sometimes

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introduced in order to limit the extent of a problem so that a numerical computation of the solution can be made feasible. This is the case with limited-area problems in oceanography and meteorology. Another such situation is the modeling of a fluid flow in an exterior domain.

In order to determine a unique solution to the problem, it is necessary to specify values of the solution at some initial time, and it is generally necessary also to impose conditions on the solution at the boundary. The problem thus becomes an initial-boundary value problem (IBVP). In some cases the correct boundary conditions can be found easily from physical considerations. At a solid wall which bounds the flow of a fluid, for example, one sets the normal component of the fluid velocity equal to zero (if effects of viscosity are to be considered, the tangential component must also vanish). In other situations the choice of boundary conditions is not as obvious. This is the case with artificial boundaries, which do not correspond to anything physical.

In general, it is necessary to be careful when choosing boundary conditions for a hyperbolic system. This can be seen most easily in the case of first-order systems in one space dimension, which we will discuss in Section 2. There we will show that various portions of the solution represent traveling waves. It will be apparent that any acceptable boundary condition must prescribe the behavior of the waves which are coming into the spatial domain, but is must not affect the waves which are leaving. Examples of suitable boundary conditions include reflection conditions, which describe incoming waves in terms of outgoing waves.

In several space dimensions the situation is more complicated than in one dimension. In this case, it is not nearly as easy to identify ''incoming'' and ''outgoing'' components of the solution. There may also be waves which move tangent to the boundary, and it may not be clear what a boundary condition should say about them. These difficulties are among the principal topics to be discussed in the present paper.

Because of the above problems, it may be difficult to determine whether a given boundary condition is suitable for a given equation in several dimensions. One approach to this question is given by the ''energy method'' (e.g., Friedrichs [4], Courant and Hilbert, vol.II [3]). This method gives criteria which are sufficient for a boundary condition to yield a well-posed problem, i.e., a problem which admits a unique solution depending continuously on the prescribed data. (See Section 3.1 for a more precise definition of ''well-posed''.) The criteria are not necessary; if the method fails to show that a boundary condition is acceptable, this may be due either to a defect in the condition or a defect in the method.

An alternate approach yielding more precise information (''normal mode analysis'') has therefore been developed. The theory includes work by Kreiss [10], Sakamoto [23], Rauch [21], Ralston [20], Majda and Osher [14], and Michelson [15] . The theory gives criteria which are essentially necessary and sufficient for a boundary condition to yield a well-posed problem.

The theory is extremely complicated and algebraic; its physical meaning tends to get buried by lengthy and detailed studies of various matrices. The principal purpose of the present paper is to examine this work from the viewpoint of wave propagation. We will show that to a great extent the theory has physical effects that one would expect from the discussion of incoming and outgoing waves given above. The concept of group velocity plays a major role in the discussion. The great complexity of the theory is due mainly to the Fourier components of the solution which correspond to waves traveling tangent to the boundary.

We now outline the contents of this paper. In Section 2 we review the standard treatment of systems in one space dimension. In Section 3 we describe the motivation which is usually given for the multi-dimensional theory. Section 3 concludes with a literature survey.

The interpretations to be given here make extensive use of the structure of the characteristic variety of the system, i.e., the high-frequency part of the ''dispersion'' relation. In Section 4 we describe this structure for some examples, and we point out that these special cases model crucial features of more general systems. For the sake of definiteness and clarity, the discussion in the remaining sections is given in terms of the examples. In Proposition 4.2 (Section 4.4) we describe a process by which incoming and outgoing modes are labelled algebraically in the theory of well-posedness.

In Section 5 we give some interpretations of the ''uniform Kreiss condition'' (U.K.C.), the algebraic criterion which characterizes admissible boundary conditions. This criterion is used to solve for ''incoming'' components in terms of ''outgoing'' components and boundary data. In this section, we also consider some weaker forms of well-posedness which are encountered when the U.K.C. is not quite satisfied. In these cases there are certain uniformities which do not hold as the frequencies of the Fourier components of the solution tend to infinity, so that certain reflection coefficients may tend to infinity.

In Section 6 we use the framework of Sections 4 and 5 to discuss some of the main ideas in the proofs by Kreiss and others which show well-posedness of the initial-boundary value problem.

Sections 4,5, and 6 form the core of this paper. The reader who is in a hurry may find it worthwhile to skim lightly over Section 2 and Section 3.2.

The theory discussed here is closely related to a theory which deals with the stability of finite difference approximations to initial-boundary value problems. The stability theory includes work by Osher [18],[19], Gustafsson, Kreiss, and Sundström [5], and Michelson [16]. Trefethen [27],[29] has recently studied this work from the viewpoint of wave propagation and has reached some conclusions which are analogous to some ideas expressed in the present paper. From time to time we will point out some similarities between the well-posedness theory and the stability theory.

2. Systems in one space dimension

We now review the situation for systems in one space dimension. In this case the problem of finding suitable boundary conditions is fairly straightforward.

We consider the system

$$(2.1) \qquad u_t = Au_x + Cu \quad ,$$

where $u(x,t)$ is a vector having n components, and $A(x,t)$ and $C(x,t)$ are $n \times n$ matrices. For simplicity, we assume that (2.1) is defined for $x > 0$ and $t > 0$. If a system is defined on a spatial domain with two boundaries, e.g., $a < x < b$, then each boundary can be treated separately in the manner to be described below.

The system (2.1) is assumed to be hyperbolic, i.e., A has real eigenvalues and a complete set of real eigenvectors. This assumption enables one to simplify the form of the system. It implies that there exists a nonsingular matrix Q such that $Q^{-1}AQ = \text{diag}\,\{\lambda_1,\ldots\lambda_n\}$. The system (2.1) can then be written as

$$(Q^{-1}u)_t = (Q^{-1}AQ)(Q^{-1}u)_x + (Q^{-1}C + Q^{-1}_t - Q^{-1}AQQ^{-1}_x)u \ ,$$

or .

$$(2.2) \qquad v_t = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} v_x + Dv \quad ,$$

where $v = Q^{-1}u$.

For the sake of simplicity, we first consider the case $D = 0$. Under this assumption each equation in (2.2) has the form

$$\frac{\partial v_j}{\partial t} - \lambda_j \frac{\partial v_j}{\partial x} = 0$$

and is an ordinary differential equation for v_j along the characteristic curves defined by $dx/dt = -\lambda_j$. (See Figure 2.1) The components v_j are constant along the corresponding characteristic curves and can thus be regarded as traveling waves which move at the characteristic velocities $dx/dt = -\lambda_j$.

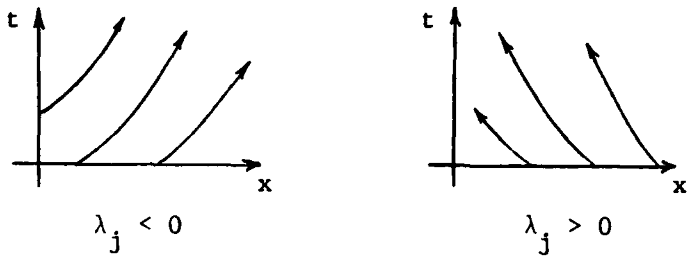


Figure 2.1. Characteristics for (2.2).

Initial values for the ordinary differential equations for v_j are provided by the values of v_j at $t = 0$ and also, when $\lambda_j < 0$, by values of v_j at the boundary $x = 0$. It is therefore necessary to prescribe an initial condition

$$v(x,0) = f(x) \quad , \quad x > 0 ,$$

where f is a given function, together with a boundary condition which defines values for the v_j corresponding to negative λ_j, i.e., the incoming components of v. It is not permissible to prescribe values for outgoing components at $x = 0$; otherwise, the boundary conditions could contradict the effect of the initial condition and thereby make it impossible for a solution to exist. The boundary conditions must therefore fit the general form

$$\text{(2.3)} \qquad v^{I}(0,t) = Sv^{II}(0,t) + g(t) \ , \ t > 0 ,$$

where v^I and v^{II} are vectors consisting of the incoming and outgoing components, respectively. Here g is a given function, and S is a rectangular matrix of appropriate dimensions which governs reflections at the boundary.

When $D \neq 0$, the equations in (2.2) are coupled together. In this case the existence of the solution can be shown via an iteration of Picard type. See, e.g., Courant and Hilbert [3]. The boundary conditions must satisfy the same criteria as before; it is necessary to prescribe values for incoming components, and it is not permissible to prescribe values for outgoing components. The identification of the incoming and outgoing components depends only on the leading order part of the system, i.e., is independent of D .

It is often of interest to know that the solution depends continuously on the prescribed data f and g. In this case the continuous dependence follows from the fact that solutions to ordinary differential equations are continuous functions of their initial data. (Also see (3.5) and the associated discussion)

3. Motivation for the multi-dimensional theory

We now begin the discussion of problems in several space dimensions. In this case it is generally not possible to construct and analyze the solution by using characteristic curves. As noted in the Introduction, additional problems may be caused by the possibility of waves moving tangent to the boundary. One can therefore expect this case to be more complicated than the previous one.

In the present section we describe the motivation which is usually given for the approach developed by Kreiss and others for studying the multi-dimensional case. In Section 3.1 we make some preliminary remarks, and in Section 3.2 we derive a condition which is necessary for the problem to be well-posed. In Section 3.3 we state a stronger condition which can be shown to be sufficient. The effects of the stronger assumptions will be discussed in later sections, mainly Section 5. The proof of sufficiency is the main source of difficulty in the theory and bears little resemblance to the derivation of the necessary condition. In Section 3.4 we survey some of the literature which deals with the subject.

3.1 Preliminaries

A fairly general form for linear hyperbolic systems is

$$u_t = Au_x + \sum_{j=1}^{m} B_j \frac{\partial u}{\partial y_j} + Cu + F \quad . \tag{3.1}$$

(See the examples in Section 4.) We consider this system for $t > 0$, and for reasons given below we assume that the spatial domain is defined by $x > 0$ and $y \in R^m$, where $m \geq 1$. In (3.1) $u(x,y,t)$ and $F(x,y,t)$ are vectors having n components, and A, B_j, and C are $n \times n$ matrices which we assume to be smooth functions of x, y, and t.

In the theory which has been developed, the system is assumed to be either strictly hyperbolic or symmetric hyperbolic. In the former case the matrices $\sigma A + \sum \omega_j B_j$ have real distinct eigenvalues for all real σ, $\omega_1, \ldots \omega_m$ for which $|\sigma| + |\omega| \neq 0$. In the latter case A and the

B_j are hermitian. (See Section 4.1 regarding ''symmetrizable'' systems.)

The spatial domain $[0,\infty) \times R^m$ has been chosen for the sake of simplicity. If one is considering a system defined on a spatial region which does not have this form but still has a smooth boundary, then one can localize the problem with a partition of unity and then map each boundary portion into the boundary of the half-space $[0,\infty) \times R^m$. In the new coordinates the problem will have the form given above.

Since (3.1) is hyperbolic, the matrix A can be assumed to be diagonal with real eigenvalues; otherwise one can find a similarity transformation which makes it diagonal and then adopt the corresponding change of dependent variable. In much of the theory A is also assumed nonsingular, i.e., the boundary $x = 0$ is noncharacteristic. (cf. Section 4.3) Unless otherwise stated, we will assume that this is the case and that the elements of A are arranged so that A has the form

$$(3.2) \qquad A = \begin{pmatrix} A^I & \\ & A^{II} \end{pmatrix}$$

where $A^I < 0$ and $A^{II} > 0$.

In analogy with the one-dimensional case, we prescribe an initial condition

$$(3.3) \qquad u(x,y,0) = f(x,y) ,$$

where f is a given function, together with a boundary condition of the form

$$(3.4) \qquad u^I(0,y,t) = S\, u^{II}(0,y,t) + g(y,t) .$$

Here g is given function, S is a rectangular matrix, and $u = ((u^I)^T, (u^{II})^T)^T$. The components of u^I and u^{II} correspond to the blocks A^I and A^{II} , respectively, in (3.2). Some remarks about the form of (3.4) are given at the end of Section 5.1.

The problem at hand is to determine whether the boundary condition (3.4) is appropriate for the system (3.1). In the one-dimensional case the answer is obvious, but in the present case it may be necessary to place some restrictions on the matrix S . The effect of the theory being described here is to identify restrictions which are necessary and sufficient for the initial-boundary value problem to be well-posed.

By ''well-posed'' we mean that for arbitrary F,f,g in suitable function spaces, the problem (3.1), (3.3), (3.4) admits a unique solution, and, furthermore, it is possible to estimate the solution in terms of F,f,g. The latter condition is equivalent to the continuous dependence of the solution on the prescribed data. A typical estimate (''energy estimate'') has the form

$$(3.5)\qquad \begin{aligned} &||u||_{\Omega x[o,t]} + ||u||_{\partial\Omega x[o,t]} + ||u(t)||_{\Omega} \\ &\leq K\,(||f||_{\Omega} + ||g||_{\partial\Omega x[o,t]} + ||F||_{\Omega x[o,t]}), \end{aligned}$$

where K is independent of u,f,F, and g (e.g., Majda and Osher [14], Rauch [21]). Ω denotes the spatial domain $x > 0$, and the norms are weighted L^2 norms or Sobolev norms on the regions indicated by the subscripts. See also the estimate (5.25).

3.2. A necessary condition for well-posedness.

We now develop the necessary condition which was mentioned in the introduction to Section 3. This condition is based on certain families of exponential solutions of the differential equation which cannot possibly satisfy the energy estimate (3.5), or any similar estimate. If these functions also satisfy the boundary conditions, then they are solutions of the initial-boundary value problem (IBVP), and the problem must be ill-posed. It is therefore necessary to guarantee that the boundary conditions exclude these functions.

The necessary condition is similar to the Godunov-Ryabenkii criterion for the stability of finite difference approximations. (See, e.g., [5],[22], or [28]).

In order to obtain the special solutions, we assume that $C = 0$, A, B_j and S are constant, and $F = g = 0$. The IBVP (3.1), (3.3), (3.4) is then

$$(3.6)\qquad \begin{aligned} &(a)\quad u_t = Au_x + \sum B_j \frac{\partial u}{\partial y_j}\quad , x > 0,\ y \in R^m\ , t > 0 \\ &(b)\quad u(x,y,0) = f(x,y) \\ &(c)\quad u^I = Su^{II}\quad , x = 0 \end{aligned}$$

The solutions of interest (''normal modes'') are elementary waves having the form

$$(3.7)\qquad u = \phi(x)\ e^{i\omega\cdot y + st}\quad ,$$

where $\omega \in R^m$ and s is complex with $\mathrm{Re}\ s > 0$. Modes of this type are associated with Laplace and Fourier transforms (see Section 4), but we will not need to use these transforms at the present time. In fact, their use would introduce unnecessary complications.

If $\det A \neq 0$, (3.6)(a) can be written as

$$(3.8)\qquad u_x = A^{-1}(u_t - \sum B_j \frac{\partial u}{\partial y_j})$$

When (3.7) is substituted into (3.8) , the result is an ordinary differential equation for the amplitude function:

$$(3.9)\qquad \begin{aligned} \phi'(x) &= A^{-1}(sI - \sum_{j=1}^{m} i\omega_j B_j)\ \phi \\ &= M(\omega,s)\phi \end{aligned}$$

The solutions of (3.9) will be discussed extensively later. For the moment it suffices to say that there are certain solutions which decay exponentially as $x \to +\infty$ and others which grow exponentially. We also note that for any solution u of the form (3.7), the related functions

$$(3.10)\qquad u_\alpha(x,y,t) = \phi(\alpha x)\ e^{i\alpha\omega\cdot y + \alpha st}$$

are also solutions of (3.6)(a), for all real α.

A necessary condition for well-posedness is implied by the following Proposition:

<u>Proposition 3.1.</u> Suppose that for some s with $\mathrm{Re}\, s > 0$ and some $\omega \in R^m$, there is a function u of the form (3.7) which satisfies the following:

(a) $\phi \neq 0$, and ϕ decays exponentially as $x \to +\infty$; and

(b) u satisfies the boundary condition (3.6)(c), i.e.,

$$u^I = Su^{II}, \text{ for } x = 0 . \tag{3.11}$$

The IBVP (3.1), (3.3), (3.4) must then be ill-posed.

<u>Proof.</u> (3.11) implies that the corresponding u_α also satisfy the boundary condition, since (3.11) is equivalent to $\phi^I(0) = S\phi^{II}(0)$. The u_α are thus solutions of the IBVP (3.6), which is a special case of (3.1), (3.3), (3.4). We also note that the u_α have finite norm with respect to x.

We now show that the family $\{u_\alpha: \alpha > 0\}$ *violates the energy* estimate (3.5). As $\alpha \to +\infty$, the solutions u_α grow at arbitrarily high exponential rates in t, since $\mathrm{Re}\, s > 0$. Equivalently, the left-hand side of (3.5) is an exponentially increasing function of α. This is not the case for the right-hand side. In this case F and g are zero, and the initial values are

$$f_\alpha(x,y) = \phi(\alpha x)\, e^{i\alpha\omega\cdot y} .$$

Sobolev norms with respect to x and y introduce derivatives which cause polynomial growth in α, but this is the most that can happen on the right-hand side. By taking α sufficiently large, we can therefore conclude that no constant K in (3.5) is adequate. The definition of well-posedness given earlier is thus violated.

Hersh [6] has pointed out that the failure of the energy estimate implies that existence and/or uniqueness must also fail to hold. His argument is based on the closed graph theorem.

A crucial part of the above argument is the fact that F and g are zero. If F and g are large whenever the solution is large, then there is no problem. But in the case considered here, it is possible for large solutions to appear without sufficient provocation from the prescribed data.

One may object that the u_α do not have finite L^2 norms or Sobolev norms with respect to y and therefore would not have to satisfy (3.5). This problem can be remedied by truncating the u_α. Let ψ be a smooth function of y which is equal to 1 for $|y| \leq 1$ and is equal to zero for $|y| \geq 2$, and consider the functions

$$u_{\alpha,\lambda}(x,y,t) = \psi(\frac{y}{\lambda})u_\alpha = \psi(\frac{y}{\lambda})\phi(\alpha x)e^{i\alpha\omega\cdot y + \alpha st}$$

for $\lambda > 0$. These functions satisfy an inhomogeneous differential equation and homogeneous boundary conditions. The failure of the energy estimate is shown by letting $\alpha,\lambda \to +\infty$. We omit the details.

The ideas in the above proof, aside from the truncation in y, have been *credited to Agmon. (cf. Kreiss [10], Agmon [1])*

The function ϕ of Proposition 3.1 is often said to be the solution of an eigenvalue problem with eigenvalue s, since it satisfies

$$\text{(3.12)} \qquad \begin{aligned} s\phi &= A\frac{d\phi}{dx} + (\sum i\omega_j B_j)\phi \\ \phi^I(0) &= S\phi^{II}(0) \\ \phi &\in L^2(0,\infty) \end{aligned}$$

The third condition may be regarded as a boundary condition at infinity. Proposition 3.1 can then be stated in the following manner:

Proposition 3.1′. The IBVP (3.1), (3.3), (3.4) is ill-posed if, for some $\omega \in R^m$, the problem (3.12) has an eigenvalue s with $\operatorname{Re} s > 0$.

We will later formulate this Proposition as an algebraic criterion which is necessary for well-posedness, but we must first characterize the solutions of the system (3.9).

The basic theory of ordinary differential equations implies that (3.9) has n linearly independent solutions $\phi_1, \ldots \phi_n$ which span the set of all solutions of (3.9). The ϕ_j can be taken to have the form

$$(3.13)(a) \quad \phi(x) = e^{\kappa x} w$$

where κ is an eigenvalue of $M(\omega,s)$ and w is a corresponding eigenvector, or

$$(3.13)(b) \quad \phi(x) = e^{\kappa x} P(x) \ ,$$

where P is a polynomial having vector coefficients. The latter form arises when κ is associated with a nontrivial Jordan block. The degree of P is less than the algebraic multiplicity of κ, and $P(0)$ is a generalized eigenvector. We will not denote explicitly the dependence of ϕ, w, and P on s and ω.

The ϕ_j give rise to exponential solutions of the partial differential equation (3.6)(a) via the relation (3.7), $u = \phi(x)e^{i\omega\cdot y + st}$. These modes will be denoted by

$$(3.14) \quad u_j(x,y,t) = \phi_j(x)e^{i\omega\cdot y + st} = \begin{cases} e^{\kappa x + i\omega\cdot y + st}\, w \\ \text{or} \quad e^{\kappa x + i\omega\cdot y + st}\, P(x) \end{cases}$$

We are interested in solutions of this type which have finite L^2 norms or Sobolev norms with respect to x on the interval $0 < x < \infty$. Information about these modes is given by the following Proposition due to Hersh [6]. Analogous properties of finite difference equations are given in Lemmas 5.1 and 5.2 of Gustafsson, Kreiss, and Sundström [5].

<u>Proposition 3.2.</u> Let ℓ denote the number of negative eigenvalues of A. If $\mathrm{Re}\ s > 0$, then the matrix $M(\omega,s) = A^{-1}(sI - \sum i\omega_j B_j)$

(see (3.9)) has ℓ eigenvalues κ with negative real part and $n - \ell$ eigenvalues with positive real part.

Proof. We first note that there are no purely imaginary eigenvalues when $\text{Re } s > 0$; if $\kappa = i\sigma$ were an eigenvalue, then

$$i\sigma w = A^{-1}(sI - \sum i\omega_j B_j)w ,$$

for some vector $w \neq 0$, and therefore

$$(i\sigma A + \sum i\omega_j B_j)w = sw .$$

But s would then have to be purely imaginary, since the system is hyperbolic.

Let $s = \eta + i\xi$. We are interested in the signs of the real parts of the eigenvalues of $M(\omega,s)$ on the domain $\eta > 0$, $\xi \in R$, $\omega \in R^m$. The eigenvalues are continuous functions of η, ξ, and ω, so the real part of each eigenvalue must have a constant sign on the domain; otherwise there would be sets on which $\text{Re } \kappa = 0$ for some κ. We can therefore count the signs at any point which may be convenient. If $\eta = 1$, $\xi = 0$, and $\omega = 0$, then $M = A^{-1}$. In this case the claimed counting is correct, and the Proposition follows.

The Proposition implies that ℓ of the functions ϕ_j in (3.13) have finite norm on the interval $0 < x < \infty$. The same is true of the solutions u_j in (3.14), for fixed y and t. Arrange indices so that these functions are $\phi_1, \ldots \phi_\ell$ and $u_1, \ldots u_\ell$.

We can now formulate an algebraic condition which is necessary for well-posedness. According to Proposition 3.1, it is necessary to prevent certain solutions u of (3.6)(a) from satisfying the homogeneous boundary condition (3.6)(c). The solutions of concern are linear combinations of $u_1, \ldots u_\ell$, and the corresponding ϕ's are linear combinations of $\phi_1, \ldots \phi_\ell$.

The boundary condition (3.6)(c) can be written as

$$[I, -S]\begin{bmatrix} u^{I} \\ u^{II} \end{bmatrix} = 0, \qquad \text{for } x = 0$$

For the solutions of interest this condition has the form

$$(3.15) \quad [I, -S][u_1, \dots u_\ell]\, c = 0 , \qquad \text{for } x = 0,$$

where c is an ℓ - vector whose components are the coefficients in the linear combination. The representation $u_j = \phi_j(x)e^{i\omega' y + st}$ (see (3.14)) implies that (3.15) is equivalent to

$$(3.16) \quad [I, -S][\phi_1(0), \dots \phi_\ell(0)]\, c = 0$$

The linear independence of $\phi_1, \dots \phi_\ell$ implies that a linear combination of $u_1, \dots u_\ell$ is zero if and only if $c = 0$. We therefore want (3.16) to have no nonzero solutions c, which means that the matrix

$$(3.17) \qquad N(\omega, s) = [I, -S][\phi_1(0), \dots \phi_\ell(0)]$$

must be nonsingular.

We should note that $N(\omega,s)$ is square; the number of rows in the boundary condition (3.6)(c) is equal to the number of negative eigenvalues of A, which is ℓ. (See (3.2), (3.4), and Proposition 3.2.) We also note that N really does depend on ω and s; according to (3.13) and the associated explanation, the vectors $\phi_1(0), \dots \phi_\ell(0)$ are eigenvectors and generalized eigenvectors of $M(\omega,s)$. They correspond to the eigenvalues which have negative real part. For an example of $N(\omega,s)$, see (5.23).

From the above discussion we can conclude the following:

Proposition 3.3. A necessary condition for well-posedness is that

$$(3.18) \qquad \det N(\omega, s) \neq 0$$

for all $\omega \in R^m$ *and all complex* s *with* $Re\ s > 0$.

3.3. The sufficient condition

We now introduce a stronger version of (3.18) which can be shown to be sufficient for well-posedness. The stronger condition is a uniform version of (3.18), and in at least many cases, it amounts to a requirement that (3.18) hold for $\text{Re } s \geq 0$, rather than just for $\text{Re } s > 0$. The effects of the stronger assumptions will be discussed in later sections.

In order to state the sufficient conditon, it will be necessary to introduce some normalizations. For any (ω, s) with $\text{Re } s > 0$, choose a basis for the ℓ - dimensional vector space of decaying solutions of the system $\Phi_x = M(\omega,s)\Phi$ (i.e.,(3.9)), and from this basis produce a new basis which is orthonormal at $x = 0$. For example, one could choose the particular functions $\Phi_1, \ldots \Phi_\ell$ discussed earlier (see (3.13) and the discussion after Proposition 3.2) and then perform the Gram - Schmidt orthogonalization process on the initial values $\Phi_1(0), \ldots \Phi_\ell(0)$. We have noted that these initial values are eigenvectors and generalized eigenvectors of $M(\omega,s)$. For reasons given below, we will not limit attention to the particular basis $\Phi_1, \ldots \Phi_\ell$.

Denote the orthonormal initial values by $\psi_1(\omega,s), \ldots \psi_\ell(\omega,s)$; and let

$$(3.19) \quad \bar{N}(\omega,s) = [I,-S][\psi_1\omega,s), \ldots \psi_\ell(\omega,s)]$$

(cf.(3.17)). The sufficient condition is the following:

Uniform Kreiss Condition (U.K.C.). There exists $\delta > 0$ such that

$$(3.20) \qquad |\det \bar{N}(\omega,s)| \geq \delta$$

for all $\omega \in R^m$ and all complex s with $\text{Re } s > 0$.

This is essentially the formulation used by Majda and Osher [14] and Michelson [15]. An alternate formulation will be discussed later. We make several comments regarding the form of (3.20):

(1) The earlier, necessary condition (3.18) did not involve any normalization of the vectors $\phi_1(0),\ldots \phi_\ell(0)$, but some sort of normalization is necessary if the uniform condition (3.20) is to hold. Otherwise, we could replace ψ_j with $\varepsilon\psi_j$, with ε small.

(2) The U.K.C. does not depend on the choice of orthonormal basis $\psi_1(\omega,s),\ldots\psi_\ell(\omega,s)$. If $\tilde{\psi}_1,\ldots \tilde{\psi}_\ell$ is another such basis, then

$$[\tilde{\psi}_1,\ldots\tilde{\psi}_\ell] = [\psi_1,\ldots\psi_\ell]Q,$$

where Q is an $\ell \times \ell$ matrix. It is easy to check that Q is unitary. Then,

$$\begin{aligned}|\det([I,-S][\tilde{\psi}_1,\ldots\tilde{\psi}_\ell])| &= \det([I,-S][\psi_1,\ldots\psi_\ell]Q)| \\ &= |\det(\bar{N}Q)| = |\det \bar{N}| \cdot |\det Q| \\ &= |\det \bar{N}|\end{aligned}$$

(3) It would suffice to impose (3.20) merely for $|\omega|^2 + |s|^2 = 1$ (and $\mathrm{Re}\ s > 0$); in each direction in the (ω,s) space it is possible to choose vectors $\psi_j(\omega,s)$ which are homogeneous of degree zero in (ω,s). For example, this can be accomplished by observing that $M(\omega,s) = A^{-1}(sI - \sum i\omega_j B_j)$ is homogeneous of degree one and that solutions of $\phi_x = M\phi$ can be written in terms of solutions for $|\omega|^2 + |s|^2 = 1$ by scaling x appropriately.

(4) The UKC applies to systems which have variable coefficients. For such systems the matrix $M(\omega,s)$ is also a function of (x,y,t). In this case the above analysis is applied to the constant-coefficient problems obtained by freezing coefficients at boundary points $(0,y,t)$, and the UKC is then required to hold uniformly in (y,t). The calculus of pseudo-differential operators is used extensively in the treatment of variable-coefficient problems. (See Section 6).

We now discuss an alternate formulation of the U.K.C. The earlier condition (3.18), $\det N(\omega,s) \neq 0$ for $\text{Re } s > 0$, is equivalent to requiring $\det \bar{N}(\omega,s) \neq 0$ for $\text{Re } s > 0$. This follows easily from a change-of-basis argument which resembles that of comment (2) above. We now show that in many cases the U.K.C. is equivalent to requiring $\det \bar{N}(\omega,s) \neq 0$ for $\text{Re } s \geq 0$, not just for $\text{Re } s > 0$. The U.K.C. is often described in this manner (e.g., in [5],[10]).

The equivalence will follow from some continuity arguments. In at least many cases (see Section 6), it is possible to choose the basis vectors $\psi_1(\omega,s),\ldots\psi_\ell(\omega,s)$ so that they are piecewise continuous functions of (ω,s). The continuity makes it possible to extend the definition of $\bar{N}(\omega,s)$ to $\text{Re } s = 0$. We note that the description of $\bar{N}(\omega,s)$ given earlier is not valid in this case; when $\text{Re } s = 0$ the system $\phi_x = M\phi$ does not, in general, have ℓ linearly independent solutions which tend to zero as $x \to +\infty$ (cf. Proposition 3.2 and Section 4.3). The equivalence mentioned above now follows from the piecewise continuity of $\bar{N}(\omega,s)$ and the compactness of the set $\{(\omega,s): |\omega|^2 + |s|^2 = 1, \ \text{Re } s \geq 0\}$.

3.4 A survey of some of the literature on the subject.

Hersh [6] studied first-order hyperbolic systems which have constant coefficients and which are defined on a half-space. He showed existence and uniqueness of solutions subject to the condition (3.18). Solutions were constructed via Fourier transforms and Laplace transforms.

Kreiss [10] used an approach which is applicable to variable-coefficient problems defined on arbitrary spatial domains having smooth boundaries. Here the main idea is to construct a ''symmetrizer'', a pseudo-differential operator which enables one to obtain the necessary energy estimate. This is the approach which will be discussed later in this paper. The construction is based on the condition (3.20). Kreiss assumed that the system is strictly hyperbolic and that the matrix A in (3.1) is nonsingular, i.e., the boundary $x = 0$ is noncharacteristic. He also assumed that the initial data are zero.

Sakamoto [23] obtained analogous results for scalar hyperbolic equations of higher order.

Ralston [20] provided an alternate method for treating some technical points in Kreiss' construction. His work extends Kreiss' results to systems having complex coefficients.

Rauch [21] considered the case of nonzero initial data and proved regularity estimates for the solution.

Majda and Osher [14] generalized the theory to the case where the boundary is ''uniformly characteristic''. This means that the matrix $A = A(x,y,t)$ in (3.1) has constant rank $< n$ for all (x,y,t) in a neighborhood of the boundary $x = 0$. They assumed that the system is symmetrizable hyperbolic.

Michelson [15] used a theory of analytic matrices to simplify some aspects of the uniformly characteristic case.

Strikwerda [25] studied ''incompletely parabolic'' systems from a point of view which is similar to that of Kreiss, et.al.

Sarason and Smoller [24] used methods of geometrical optics to study the behavior of hyperbolic systems in spatial regions with corners. They considered the case of rays which are not tangent to the boundary.

Majda [13] used Kreiss symmetrizers to study the linearized stability of multi-dimensional shock fronts.

The well-posedness theory discussed here is closely related to a theory for the stability of finite difference approximations to initial-boundary value problems for hyperbolic systems. Gustafsson, Kreiss, and Sundström [5] showed that a condition analogous to the U.K.C. implies the stability of difference approximations in one space dimension. Osher [18], [19] had earlier used different techniques to obtain stability results for a restricted class of methods. Michelson [16] has obtained results for dissipative approximations in several dimensions when the boundary is noncharacteristic and the initial data are zero. A more extensive survey of the literature is given by Trefethen [27].

4. Characteristic variety; examples.

In this section we discuss various matters related to the ''characteristic variety'' of the system (3.1). The discussion will give useful information about the nature of the well-posedness theory and its physical interpretation.

Before we proceed, we must perform some transformations on the IBVP (3.1), (3.3), (3.4). Through a process described at the beginning of Section 5.2, the effects of the initial data can be incorporated into the forcing term F. We thus assume $f = 0$. We also assume that $C = 0$ and that the coefficients are constant. (See the comments after (4.8)) We now apply a Fourier transform in the tangent variable y, with dual variable $i\omega$, and a Laplace transform in t, with dual variable $s = \eta + i\xi$ $(\eta > 0)$. The latter transform is defined by

$$Lw(s) = \int_0^\infty e^{-st} w(t)\, dt. \tag{4.1}$$

Under the above transformations the system (3.1) becomes

$$s\hat{u}(x,\omega,s) = A\hat{u}_x + \sum_{j=1}^{m} i\omega_j B_j \hat{u} + \hat{F}(x,\omega,s)$$

Here we used the fact $f = 0$. The IBVP can then be written as

$$\text{(4.2)} \qquad \begin{aligned} &\text{(a)} \quad \hat{u}_x = M(\omega,s)\hat{u} - A^{-1}\hat{F} \quad , \; x > 0 \\ &\text{(b)} \quad \hat{u}^{I}(0,\omega,s) = S\hat{u}^{II} + \hat{g}(\omega,s), \end{aligned}$$

where

$$M(\omega,s) = A^{-1}(sI - \sum_j i\omega_j B_j) \tag{4.3}$$

It may not be apparent why one would use transforms in (y,t) rather than, for example, in (x,y). This question will be discussed at the beginning of Section 4.3.

For later reference, we show that the use of Laplace and Fourier transforms enables one to express solutions of (3.1) as superpositions of elementary wave-like solutions. The Laplace transform (4.1) of a function w is the same as the Fourier transform of the function

$$w_o(t) = \begin{cases} e^{-\eta t} w(t) & , t > 0 \\ 0 & , t < 0 \end{cases}$$

The factor $e^{-\eta t}$ is inserted to ensure integrability. The application of an inverse Fourier transform shows that $w(t)$ is a superposition of modes $e^{\eta t} e^{i\xi t} = e^{st}$, where η is fixed. The Fourier transform in y introduces an additional oscillatory factor $\exp(i\omega \cdot y)$. Furthermore, the solutions of the inhomogeneous system (4.2)(a) are based upon factors $\exp(\kappa x)$ for various eigenvalues κ of $M(\omega,s)$. (See Section 5.2) The solutions of the partial differential equation are thus built up from modes

$$e^{\kappa x + i\omega \cdot y + st} . \tag{4.4}$$

These modes will be discussed extensively later.

The normal modes (4.4) were also used in the discussion of the necessary condition in Section 3. However, we deliberately avoided superimposing such modes in order to avoid unnecessary complications.

We now describe the main topics to be discussed in the present section.

__Definition__ . The __characteristic__ __variety__ of the system (3.1) is the set of all $\sigma \in \mathbb{R}$, $\omega \in \mathbb{R}^m$, $\xi \in \mathbb{R}$ such that

$$\det[\xi I - (\sigma A + \sum_j \omega_j B_j)] = 0 \tag{4.5}$$

The matrix function $\sigma A + \sum \omega_j B_j$ is commonly called the ''principal symbol'' of (3.1). Some remarks in Section 3.1 imply that if the system is hyperbolic, then the symbol has real eigenvalues and a complete set of eigenvectors. In general, A and the B_j may depend on (x,y,t), and thus the characteristic variety may also depend on (x,y,t). A typical relation (4.5) is graphed in Figure 4.1 in Section 4.2.

There are a couple of reasons why the relation (4.5) is relevant to the study of well-posedness. First, a topic of major interest in the theory is the structure of the matrix $M(\omega,s)$ in (4.3). For reasons to be discussed later, a particularly important topic is the behavior of $M(\omega,s)$ as $\mathrm{Re}\ s \to 0$. This is already suggested by the manner in which the necessary condition (3.18) is strengthened by the U.K.C. (3.20). For the moment let us consider the limiting case $M(\omega,i\xi)$. If $i\sigma$ is a purely imaginary eigenvalue and z is a corresponding eigenvector, then

(4.6)(a) $$i\sigma z = A^{-1}(i\xi - \sum i\omega_j B_j)z,$$

or

(4.6)(b) $$\xi z = (\sigma A + \sum \omega_j B_j)z.$$

The point (σ,ω,ξ) must satisfy the relation (4.5). The characteristic variety will thus enable one to visualize the behavior of $M(\omega,s)$ as $\mathrm{Re}\ s \to 0$. The relations (4.6) may be compared with the conclusions of Proposition 3.2.

A second reason for studying the characteristic variety is that it is related to the physical phenomenon of wave propagation. Suppose that (3.1) has constant coefficients and that $C = 0$ and $F = 0$, so that (3.1) is

(4.7) $$u_t = Au_x + \sum B_j \frac{\partial u}{\partial y_j}$$

If a plane wave solution

(4.8) $$z\, e^{i(\sigma x + \omega \cdot y + \xi t)}$$

is inserted into (4.7), where z is a vector, the result is (4.6)(b). The characteristic variety thus describes the set of all wave numbers and frequencies of plane wave solutions of (4.7).

This analysis is approximately valid in more general cases if the frequencies are sufficiently large. When $C \neq 0$, one obtains the equation

$$\xi z = (\sigma A + \sum \omega_j B_j)z + Cz$$

instead of (4.6)(b), but the effects of C are small for large σ, ω, ξ. Also, in the case of variable coefficients the coefficients appear nearly constant to a wave whose frequencies are sufficiently high.

One may object that the mode (4.8) does not satisfy the homogeneous initial condition used to obtain (4.2)(a). This issue is settled by the discussion appearing in Section 5.2.

In the theory of well-posedness one actually encounters the modes (4.4) rather than the modes (4.8). However, the relations between the two cases are of major interest and will be discussed extensively later. (See Proposition 4.2 and Section 5.3)

We now outline the contents of the remainder of this section. In Section 4.1 we give some examples of first-order hyperbolic systems. In Section 4.2 we discuss the structure of the characteristic variety for systems like those mentioned in Section 4.1. In Section 4.3 we use the characteristic variety to describe some properties of $M(\omega, i\xi)$, and we point out that the examples model some crucial aspects of the behavior of more general systems. A major point of interest in Sections 4.2 and 4.3 is the problem of identifying the incoming and outgoing portions of the solution. In Section 4.4 we discuss the manner in which these portions are labelled algebraically by certain processes appearing in the theory of well-posedness. Some of the content of Sections 4.2 and 4.3 has already appeared in [7].

4.1 <u>Examples of first-order hyperbolic systems</u>

In this sub-section we list the shallow water equations, the two-dimensional isentropic Euler equations of gas dynamics, and Maxwell's equations.

The shallow water equations can be written in the form

$$
(4.9) \qquad \begin{aligned} & u_t + uu_x + vu_y + \phi_x = 0 \\ & v_t + uv_x + vv_y + \phi_y = 0 \\ & \phi_t + (\phi u)_x + (\phi v)_y = 0 \end{aligned}
$$

(For a derivation see, e.g., Courant and Friedrichs [2], Whitham [31].) This system describes the motion of an incompressible fluid when the fluid is bounded below by a solid boundary and its depth is small relative to a typical horizontal length scale. In (4.9) $u(x,y,t)$ and $v(x,y,t)$ are the horizontal components of velocity and ϕ is the geopotential; i.e., $\phi = gh(x,y,t)$, where g is the acceleration due to gravity and h is the height of the free surface. Coriolis effects can be included in (4.9) by adding the terms $-fv$ and fu to the left sides of the first and second equations, respectively.

The system (4.9) can be written as

$$
(4.10) \qquad q_t + \begin{pmatrix} u & 0 & 1 \\ 0 & u & 0 \\ \phi & 0 & u \end{pmatrix} q_x + \begin{pmatrix} v & 0 & 0 \\ 0 & v & 1 \\ 0 & \phi & v \end{pmatrix} q_y = 0,
$$

where $q = (u,v,\phi)^T$. If we pre-multiply (4.10) by the diagonal matrix diag $\{\phi,\phi,1\}$, the result has the form

$$
A_o q_t + A_1 q_x + A_2 q_y = 0,
$$

where the A_j are symmetric and A_o is positive definite. (4.10) is therefore a ''symmetrizable'' hyperbolic system (e.g., Friedrichs [4], John [9]).

The system is nonlinear and thus not covered by the well-posedness theory which is the subject of the present paper. However, the theory is applicable to the linearized version of the system. To obtain this version, suppose that $q = (u,v,\phi)^T$ and $q + q' = (u + u', v + v', \phi + \phi')^T$ are solutions of (4.10), and suppose that the perturbations u',v',ϕ' are small. If we substitute $q + q'$ into (4.10) and neglect terms which are quadratic in the perturbation, the result is the following linear system for q':

$$(4.11) \quad q'_t + \begin{pmatrix} u & 0 & 1 \\ 0 & u & 0 \\ \phi & 0 & u \end{pmatrix} q'_x + \begin{pmatrix} v & 0 & 0 \\ 0 & v & 1 \\ 0 & \phi & v \end{pmatrix} q'_y + Dq' = 0$$

The matrix D involves various derivatives of u, v, p; and if the system (4.9) includes Coriolis effects then the parameter f would also appear in D.

We now transform (4.11) to a symmetric system. We first note that (4.11) is symmetrizable in the same sense that the nonlinear system (4.10) is symmetrizable. This would yield the form (3.1) discussed earlier, with A and the B_j symmetric, except that the time derivative would have a coefficient matrix which is not the identity. In order to avoid this coefficient, we introduce a change of variable. Let $Q = \text{diag}\,\{\sqrt{\Phi}, \sqrt{\Phi}, 1\}$, and note that

$$Q \begin{pmatrix} u & 0 & 1 \\ 0 & u & 0 \\ \phi & 0 & u \end{pmatrix} Q^{-1} = \begin{pmatrix} u & 0 & \sqrt{\Phi} \\ 0 & u & 0 \\ \sqrt{\Phi} & 0 & u \end{pmatrix}$$

A similar conclusion can be reached about the coefficient of q'_y. The system (4.11) can therefore be transformed to

$$(4.12) \quad w_t + \begin{pmatrix} u & 0 & \sqrt{\Phi} \\ 0 & u & 0 \\ \sqrt{\Phi} & 0 & u \end{pmatrix} w_x + \begin{pmatrix} v & 0 & 0 \\ 0 & v & \sqrt{\Phi} \\ 0 & \sqrt{\Phi} & v \end{pmatrix} w_y + Ew = 0,$$

where $w = Qq' = (u'\sqrt{\Phi}, v'\sqrt{\Phi}, \phi')$. The coefficient E is different from the coefficient D in (4.11) because of some derivatives of Q which are encountered during the change of variable. The quantity $\sqrt{\Phi} = \sqrt{gh}$ is the speed of gravity waves in shallow water and will be denoted below by c.

The two-dimensional isentropic Euler equations of gas dynamics have a structure which is similar to that of the shallow water equations. The Euler equations are

$$
\begin{aligned}
& u_t + uu_x + vu_y = -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
(4.13) \qquad & v_t + uv_x + vv_y = -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
& \rho_t + (\rho u)_x + (\rho u)_y = 0 \\
& \qquad\qquad p = p(\rho)
\end{aligned}
$$

Here $u(x,y,t)$ and $v(x,y,t)$ are the x- and y-components of velocity, respectively; ρ is the density of the fluid; and p is the pressure. (4.13) can also be written as

$$
(4.14) \qquad q_t + \begin{pmatrix} u & 0 & \frac{1}{\rho}\frac{dp}{d\rho} \\ 0 & u & 0 \\ \rho & 0 & u \end{pmatrix} q_x + \begin{pmatrix} v & 0 & 0 \\ 0 & v & \frac{1}{\rho}\frac{dp}{d\rho} \\ 0 & \rho & v \end{pmatrix} q_y = 0
$$

where $q = (u,v,\rho)^T$. The system (4.14) can be symmetrized by pre-multiplying by the diagonal matrix diag $[\rho,\rho,\frac{1}{\rho}\frac{dp}{d\rho}]$. If $\frac{dp}{d\rho} > 0$ this matrix is positive definite, so that (4.14) is symmetrizable hyperbolic. From now on we will assume $\frac{dp}{d\rho} > 0$; in many situations $p = K\rho^{\gamma}$, where γ and K are constants and $\gamma > 1$.

The system (4.14) can be linearized and then symmetrized to produce a form like that of (4.12). This process yields

$$
(4.15) \qquad w_t + \begin{pmatrix} u & 0 & c \\ 0 & u & 0 \\ c & 0 & u \end{pmatrix} w_x + \begin{pmatrix} v & 0 & 0 \\ 0 & v & c \\ 0 & c & v \end{pmatrix} w_y + Ew = 0
$$

Here $w = (u', v', (\frac{c}{\rho})\rho')^T$, where u' and v' are the perturbations in the velocity components, ρ' is the perturbation in the density, and $c = \sqrt{\frac{dp}{d\rho}}$. The latter quantity is the local speed of sound.

The above discussion implies that the linearized shallow water equations and the linearized Euler equations can each be written in the form (3.1) discussed earlier,

$$(4.16) \qquad w_t = Aw_x + Bw_y + Cw,$$

where in this case

$$A = -\begin{pmatrix} u & 0 & c \\ 0 & u & 0 \\ c & 0 & u \end{pmatrix}, \quad B = -\begin{pmatrix} v & 0 & 0 \\ 0 & v & c \\ 0 & c & v \end{pmatrix},$$

and $C = -E$. An analogous form can be obtained for the Euler equations in three dimensions.

For later reference we discuss briefly the relation (4.5) for the system (4.16). The principal symbol of (4.16) is $\sigma A + \omega B$, and its eigenvalues are

$$(4.17) \qquad \begin{aligned} \xi_1 &= -(u,v)\cdot(\sigma,\omega) = -u\sigma - v\omega \\ \xi_2, \xi_3 &= -(u,v)\cdot(\sigma,\omega) \pm c(\sigma^2 + \omega^2)^{1/2} \end{aligned}$$

A typical relation (4.17) is graphed in Figure 4.1. The eigenvectors corresponding to the eigenvalues ξ_1, ξ_2, ξ_3 are

$$(4.18) \qquad \begin{aligned} z_1 &= (-\omega, \sigma, 0)^T \\ z_2, z_3 &= (\pm\sigma, \pm\omega, (\sigma^2 + \omega^2)^{1/2})^T \end{aligned}$$

We now discuss Maxwell's equations of electromagnetism. These are

$$(4.19)\quad \begin{aligned}&(a)\quad \operatorname{div} E = \frac{\rho}{\varepsilon_o}\\ &(b)\quad \operatorname{div} B = 0\\ &(c)\quad \frac{\partial E}{\partial t} = c^2(\operatorname{curl} B - \mu_o J)\\ &(d)\quad \frac{\partial B}{\partial t} = -\operatorname{curl} E,\end{aligned}$$

where E is the electric field intensity, B is the magnetic induction, ρ is the total charge density, J is the current density, c is the speed of light, and ε_o and μ_o are constants. E,B and J are vectors having three components.

In certain situations the system (4.19) can be simplified by deleting the first two equations. The reason is the following (e.g., Lorrain and Corson [12]). If we take the divergence of (d), the result is $\frac{\partial}{\partial t}(\operatorname{div} B) = -\operatorname{div}\operatorname{curl} E = 0$. If at each point in space $\operatorname{div} B = 0$ at some time, then (b) follows from (d) and may be omitted. Similarly, equation (c) implies $\frac{\partial}{\partial t}(\operatorname{div} E) = -c^2\mu_o \operatorname{div} J = c^2\mu_o\, \partial\rho/\partial t = \varepsilon_o^{-1}\, \partial\rho/\partial t$, since $c = (\varepsilon_o\mu_o)^{-1/2}$. Thus $\operatorname{div} E$ and $\frac{\rho}{\varepsilon_o}$ differ by a constant at each point in space. If for each point they are simultaneously zero at some time, then equation (a) is a consequence of (c) and may also be eliminated from the system.

Under the above conditions Maxwell's equations reduce to the form

$$(4.20)\quad \frac{\partial}{\partial t}\begin{pmatrix}E\\B\end{pmatrix} = \begin{pmatrix}0 & c^2\operatorname{curl}\\ -\operatorname{curl} & 0\end{pmatrix}\begin{pmatrix}E\\B\end{pmatrix} + F$$

We now show that this system is hyperbolic. If the spatial variables are denoted by $(x,y) = (x,y_1,y_2)$, as in (3.1), then the curl operator in (4.20) has the representation

$$\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\frac{\partial}{\partial y_1} + \begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\frac{\partial}{\partial y_2}$$

It follows that (4.20) is symmetric hyperbolic if $c = 1$. If $c \neq 1$ the symmetric form can be obtained by replacing E with $c^{-1}E$ and t with ct.

The principal symbol of (4.20) will be of interest in later discussions. A calculation shows that its eigenvalues all have multiplicity two and are given by

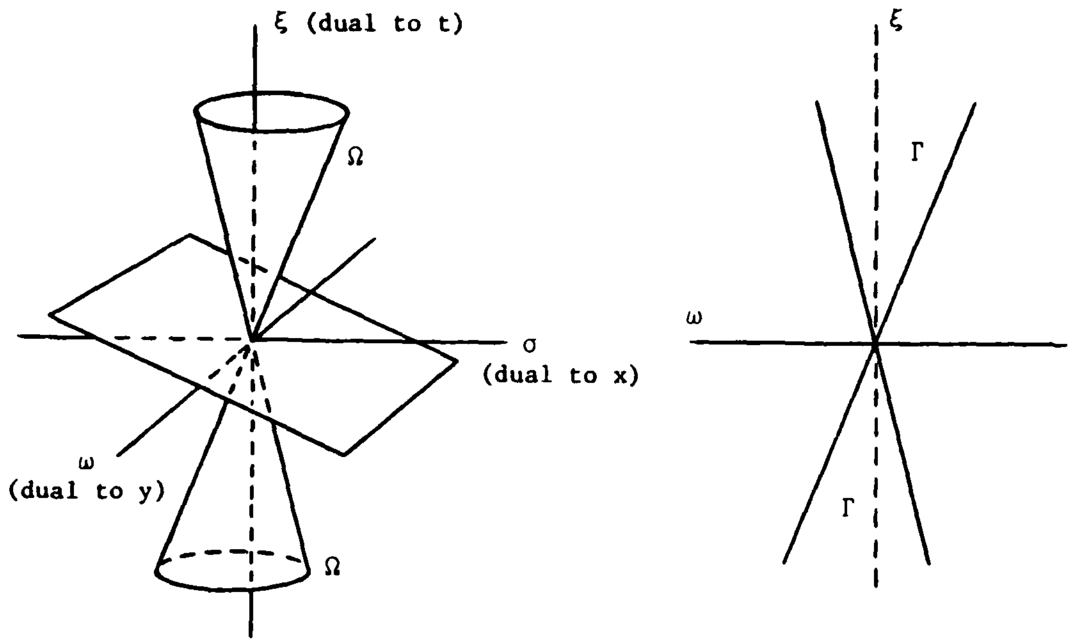
$$(4.21) \qquad \begin{aligned} &\xi_1,\ \xi_2 = 0 \\ &\xi_3,\ \xi_4 = c(\sigma^2 + |\omega|^2)^{1/2} \\ &\xi_5,\ \xi_6 = -c(\sigma^2 + |\omega|^2)^{1/2} \end{aligned}$$

Here $\sigma, \omega_1, \omega_2$ are dual to x, y_1, y_2, respectively, and $|\omega|^2 = \omega_1{}^2 + \omega_2{}^2$. The components of the corresponding eigenvectors are rational functions of the quantities $\sigma, \omega_1, \omega_2$ and $(\sigma^2 + |\omega|^2)^{1/2}$.

4.2 Structure of the characteristic variety; group velocity.

In this section we discuss the structure of the characteristic variety and how this structure is related to the problem of identifying the incoming and outgoing portions of the solution. As suggested in Section 1, the latter question can be of fundamental importance in the study of well-posedness of initial-boundary value problems.

A typical relation (4.5) is graphed in Figure 4.1. This is the characteristic variety for the linearized shallow water equations and the linearized two-dimensional Euler equations. (See (4.17)) The plane corresponds to the eigenvalue ξ_1, and the cones correspond to ξ_2 and ξ_3. The cones are right circular cones if and only if the plane is horizontal. We let Ω denote the double cone and Γ denote the set of all (ω, ξ) which correspond to points on Ω.



(a) Characteristic variety **(b) Projection onto (ω,ξ) space**

Figure 4.1

The characteristic variety for Maxwell's equations (4.20) has a similar structure. In this case each determination of ξ is a function of three variables $(\sigma,\omega_1,\omega_2)$, and each such ξ has multiplicity two. The smallest eigenvalue is zero and would thus correspond to a horizontal plane in Figure 4.1(a).

As noted in the introduction to Section 4, the relation (4.5) describes the set of all (σ,ω,ξ) in wave-like solutions

$$e^{i(\sigma x + \omega\cdot y + \xi t)} \tag{4.22}$$

of the system (4.7), which is a special case of (3.1). In the current discussion we are particularly interested in the directions in which these waves propagate. In order to deal with this question, we first make some remarks about the concept of group velocity as it applies to the present situation. We then show how incoming and outgoing portions of the solution can be identified directly from graphs like Figure 4.1(a).

This discussion will be a description of solutions of (3.1) when $C = 0$ and the coefficients are constant ; and for sufficiently high frequencies it will be an approximate description of the general case, where C may be nonzero and the coefficients may depend on (x,y,t). (See the introduction to Section 4.) At the end of this sub-section we relate this discussion to the theory of propagation of singularities, which provides a rigorous treatment of the more general case.

For each fixed t, the individual mode (4.22) is constant along lines in the (x,y) plane for which $(x,y)\cdot(\sigma,\omega)$ is constant. It then follows that the wave propagates in the direction (σ,ω) with phase speed $-\xi(\sigma^2 + |\omega|^2)^{-1/2}$. However, when one superimposes various waves of the form (4.22), the net effect is to produce a solution in which energy is propagated with a velocity which may be different from the phase velocities of the individual waves. This velocity is the group velocity, and in the notation of (4.22) it is given by

$$(4.23) \qquad \operatorname{grad}(-\xi) = \left(-\frac{\partial\xi}{\partial\sigma} \quad -\frac{\partial\xi}{\partial\omega}\right)$$

Group velocity and phase velocity differ, for example, in wave groups which propagate in one dimension and which are "dispersive", i.e., the phase velocity varies with wavelength. (e.g., Whitham [31]) In multi-dimensional problems the two velocities may also differ for reasons which are not related to variations in wavelength. This effect can be seen most easily in the case of the simple equation

$$(4.24) \qquad u_t + u_x + u_y = 0$$

The solutions to this equation are constant along characteristic curves for which $\frac{dx}{dt} = \frac{dy}{dt} = 1$, and they thus represent translations in the (1,1) direction. However, when one substitutes a wave form (4.22) into (4.24), the result is $\xi = -\sigma-\omega$, and by choosing (σ,ω) appropriately one can obtain a wave-like solution having a phase velocity which points in any desired direction. This direction cannot be meaningful because of the translatory nature of (4.24). In fact, suppose that a family of such waves is superimposed by the integral

$$\int e^{i(\sigma x + \omega y - (\sigma+\omega)t)} \hat{f}(\sigma,\omega)\, d\sigma d\omega,$$

where $\hat{f}$ is an amplitude function. This integral can be written

$$\int e^{i[\sigma(x-t) + \omega(y-t)]} \hat{f}(\sigma,\omega)\, d\sigma\, d\omega$$

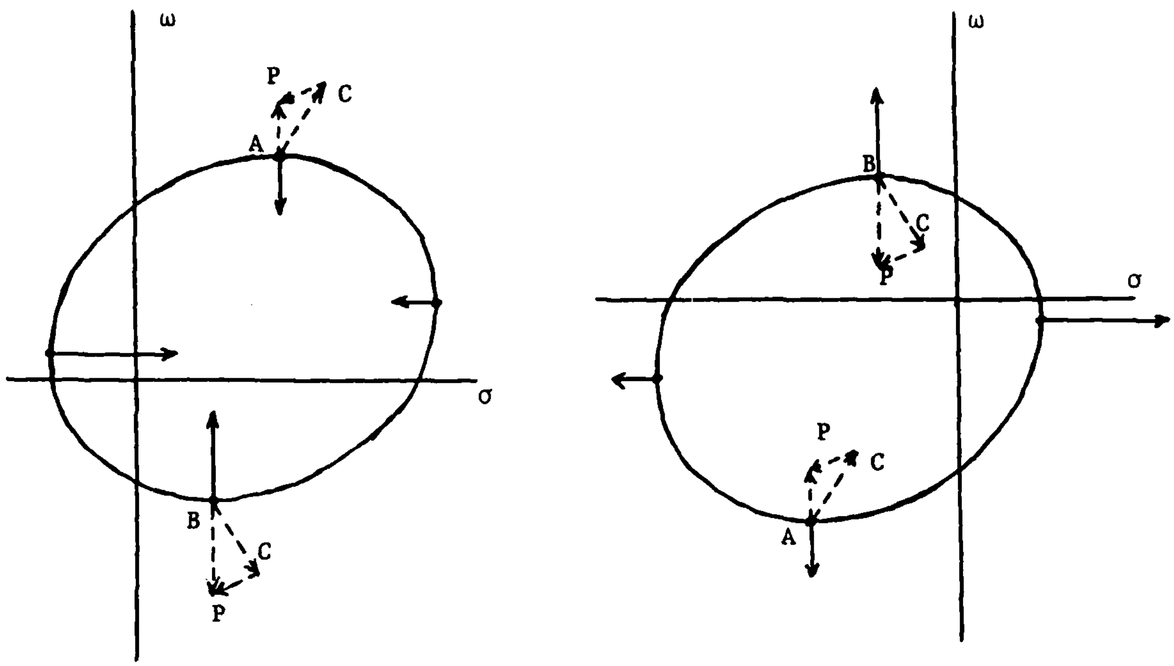
$$= f(x - t,\ y - t),$$

where f is the inverse Fourier transform of $\hat{f}$. The individual waves can move with phase velocities having any direction whatever, but the waves superpose to produce a simple translatory effect. In this case the group velocity is $(-\frac{\partial\xi}{\partial\sigma}, -\frac{\partial\xi}{\partial\omega}) = (1,1)$, which is the direction of translation.

Some interesting applications of the concept of group velocity are found in the study of finite difference approximations. See, e.g., Trefethen [27], [28] and Vichnevetsky and Bowles [30].

We now relate the directions of the group velocity to the structure of the characteristic variety graphed in Figure 4.1(a). In this particular example there are three determinations of ξ as a function of (σ,ω). The determination corresponding to the plane is simplest to analyze; its gradient is constant, so the group velocity is a constant function of (σ,ω). The plane thus represents a translatory motion which is similar to that found in solutions of the equation $u_t + u_x + u_y = 0$. In the example illustrated here the group velocity points into the spatial domain $x > 0$.

The cones Ω are more complicated than the plane. In Figure 4.2 we show cross-sections of these cones for fixed $\xi > 0$ and fixed $\xi < 0$. The directions of group velocity are indicated by the solid arrows; the group velocity is grad $(-\xi) = (-\frac{\partial\xi}{\partial\sigma}, \frac{\partial\xi}{\partial\omega})$, so the direction of this velocity is opposite the direction of most rapid increase in $\xi = \xi(\sigma,\omega)$. These directions are orthogonal to the level sets shown in the Figure. In this picture the group velocities are different from the phase velocities, since the latter always point along lines which pass through the origin.



(a) $\xi > 0$ (b) $\xi < 0$

Figure 4.2. Cross-sections of Ω for fixed ξ

In parts (a) and (b) of Figure 4.2 the points A,B denote the points (σ, ω, ξ) corresponding to group velocity which is tangent to the boundary $x = 0$. The relevance of these points will be discussed below and in the next sub-section. We now explain their locations. The vertical coordinates of points on Ω are given by sums of vertical coordinates of the plane and those of a right circular cone. (See (4.17)) At each point A or B the gradient of the right circular cone is denoted by the dotted arrow C, which points along a line passing through the origin. The gradient of the plane is indicated by the dotted arrow P. Their sum lies along the same line as the group velocity.

The cones Ω correspond to the propagation of sound waves and gravity waves, respectively, for the linearized Euler equations and the linearized shallow water equations. In the analogous picture for Maxwell's equations Ω would correspond to the propagation of electromagnetic waves.

We now mention some algebraic aspects of the discussion given above. We have denoted by Γ the set of all (ω,ξ) corresponding to points on Ω i.e., Γ is the projection of Ω onto the (ω,ξ) space. For the example illustrated here the cones Ω determine two values of σ for each (ω,ξ) in the interior of Γ. The above discussion implies that each determination of $\sigma = \sigma(\omega,\xi)$ on each component of Γ can be associated with motion into or out of the spatial domain $x > 0$. Furthermore, some remarks made in the introduction to Section 4 imply that for each such σ, the number $i\sigma$ is an eigenvalue of $M(\omega,i\xi)$. (See (4.6)(a)) This suggests that one can identify incoming and outgoing portions of the solution by diagonalizing $M(\omega,i\xi)$. This subject will be addressed in the next sub-section. As part of this discussion we point out that $M(\omega,i\xi)$ is defective when (ω,ξ) corresponds to tangential incidence. In Section 4.4 we describe a process by which the incoming and outgoing portions of the solution are labelled algebraically in the theory of well-posedness.

We now relate the earlier discussion of group velocity to the theory of propagation of singularities. The latter is mainly a discussion of the propagation of high-frequency portions of the solution, since the smoothness of a function is governed by the rate of decay of its Fourier transform.

Denote (x,y,t) by z and (σ,ω,ξ) by ζ, and let $p(z,\zeta)$ denote the determinant in (4.5). The singularities in the solution propagate along bicharacteristic curves, which are curves $(z(\tau),\zeta(\tau))$ satisfying the Hamilton-Jacobi equations

$$(4.25)\qquad \begin{aligned} &(a) \quad \dot{z} = \operatorname{grad}_{\zeta}\, p(z,\zeta) \\ &(b) \quad \dot{\zeta} = -\operatorname{grad}_{z}\, p(z,\zeta) \end{aligned}$$

and the constraint $p(z,\zeta) = 0$. (e.g., Nirenberg [17], Taylor [26]) The curve $z(\tau)$ denotes the path of propagation in the (x,y,t) space, and the curve $\zeta(\tau)$ records any frequency shifts which take place during the propagation. For a discussion of ray theory in a more applied setting, see, e.g., Whitham [31].

Equation (4.25)(a) says that the direction $\dot{z}$ is parallel to $\text{grad}_\zeta\ p(z,\zeta)$, i.e., is orthogonal to the level set $\{\zeta : p(z,\zeta) = 0\}$. This set is the characteristic variety defined by (4.5); an example is graphed in Figure 4.1. It then follows that the projection of $\dot{z}$ onto the spatial variables (x,y) is perpendicular to level sets of the type graphed in Figure 4.2. That is, the direction of propagation lies along the same line as the group velocity. One can check easily that the direction of propagation is the direction of the group velocity, rather than the exact opposite; for example, on the upper cone ($\xi > 0$) in Figure 4.1, an increase in t corresponds to a movement of (x,y) toward the center of the cone.

4.3 Properties of $M(\omega, i\xi)$

In this sub-section we use the structure of the characteristic variety to describe some properties of the matrix $M(\omega,i\xi) = A^{-1}(i\xi I - \sum i\omega_j B_j)$. (See (4.3)) We will deal mainly with the diagonalizability of this matrix and the occurrence of purely imaginary eigenvalues. A noted in the previous sub-section, the behavior of $M(\omega,i\xi)$ is closely related to the problem of identifying the incoming and outgoing portions of the solution.

The matrix $M(\omega,s)$ arises from the use of Fourier and Laplace transforms with respect to y and t, respectively. Before we proceed with the discussion of $M(\omega,i\xi)$, we first mention why one would transform in (y,t) rather than in some other set of variables.

First, there can be a problem with transforming in the normal variable x. Such a transformation would require information about the solution away from the boundary $x = 0$, and this does not seem appropriate in a discussion of boundary conditions. This difficulty can be avoided by transforming in (y,t).

A second reason is related to the problem of identifying incoming and outgoing modes. We have already noted that the eigenvalues $i\sigma = i\sigma(\omega,\xi)$ of $M(\omega,i\xi)$ are very useful in this regard. On the other hand, if we were to transform (3.1) with respect to (x,y), say, then the result would be

$$(4.26) \qquad \hat{u}_t(\sigma,\omega,t) = (i\sigma A + \sum i\omega_j B_j)\hat{u}$$

(if $C = 0$, $F = 0$). The properties of solutions of this system are governed by the eigenvalues $i\xi$ of $i\sigma A + \sum i\omega_j B_j$. Unfortunately, the discussion in Section 4.2 implies that various incoming and outgoing modes are mixed together in certain determinations of $i\xi$ as a function of (σ,ω). These determinations are associated with the cones Ω. It thus does not seem possible to accomplish the desired separation by studying the transformed equation (4.26).

We now discuss the eigenvalues of $M(\omega,i\xi)$. We recall that if $i\sigma$ is a purely imaginary eigenvalue and z is a corresponding eigenvector, then

$$(4.27) \qquad i\sigma z = A^{-1}(i\xi I - \sum_j i\omega_j B_j)z,$$

or

$$(4.28) \qquad i\xi z = (i\sigma A + \sum i\omega_j B_j)z$$

According to remarks made in the introduction to Section 4, the symbol

$$(4.29) \qquad i\sigma A + \sum_j i\omega_j B_j$$

has purely imaginary eigenvalues and a complete set of real eigenvectors. This fact will be exploited in Proposition 4.1.

In general, the number of purely imaginary eigenvalues of $M(\omega,i\xi)$ may vary with the position of (ω,ξ). In the case of the shallow water or Euler equations there are two eigenvalues $i\sigma$ associated with the double cone Ω, when (ω,ξ) is in the interior of Γ. (See Figures 4.1 and 4.2) As (ω,ξ) approaches the edge of Γ, these eigenvalues coalesce, and as (ω,ξ) leaves Γ these eigenvalues leave the imaginary axis. They cannot be purely imaginary, since for any imaginary eigenvalue $i\sigma$ the point (σ,ω,ξ) must lie on one of the surfaces in Figure 4.1.

In fact, for (ω,ξ) outside Γ these eigenvalues must have the form $\pm\rho + i\sigma$, where ρ is real. The eigenvalues of the real matrix

$$-iM(\omega, i\xi) = A^{-1}(\xi I - \sum \omega_j B_j)$$

are either real or come in complex conjugate pairs $\sigma \pm i\rho$. When we multiply by i the result is $\pm\rho + i\sigma$.

For the Euler and shallow water equations there is also a value $i\sigma$ associated with the plane illustrated in Figure 4.1. In many cases there is no difficulty with expressing σ as a function of (ω,ξ). The plane satisfies the equation

$$\xi = -\sigma u - \omega v$$

(see (4.17)), so $i\sigma$ can be written in terms of (ω,ξ) whenever $u \neq 0$. This condition is equivalent to requiring $\det A \neq 0$, where A is the coefficient of w_x in (4.16). In Section 3.1 we assumed that this condition was satisfied, and we used it to write $M(\omega,s) = A^{-1}(sI - \sum i\omega_j B_j)$ (e.g., (3.9)).

The assumption $\det A \neq 0$ has a physical interpretation. The vector field (u,v) associated with the system (4.16) is the velocity of the flow about which the system has been linearized. If A is to be nonsingular, then this flow cannot be zero and cannot be tangent to the boundary $x = 0$. In Figure 4.1(a) the plane cannot be horizontal and cannot have a gradient which is parallel to the ω-axis. The assumption $\det A \neq 0$ is thus not always valid for the shallow water and Euler equations.

In addition, it is never valid for Maxwell's equations. This can be seen from the representation (4.20) for the system or from the formula (4.21).

We now make some remarks about the diagonalizability of $M(\omega, i\xi)$. Suppose that the characteristic variety has a cone Ω, and suppose that for (ω,ξ) in Γ there is no difficulty with solving for values of $i\sigma$ associated with surfaces other than Ω. There will be no need to make any assumptions regarding the multiplicities of any of the eigenvalues. The following Proposition may be contrasted with Proposition 3.2, which describes $M(\omega,s)$ for $\mathrm{Re}\ s > 0$.

Proposition 4.1. If (ω,ξ) is in the interior of Γ, then $M(\omega,i\xi) = A^{-1}(i\xi I - \sum i\omega_j B_j)$ has purely imaginary eigenvalues and a complete set of real eigenvectors. This is not the case if (ω,ξ) is outside Γ. The eigenvectors can be determined from those of the symbol (4.29), $i\sigma A + \sum i\omega_j B_j$.

Proof. Equations (4.27) and (4.28) show that the eigenvectors of (4.29) are also eigenvectors of $M(\omega,i\xi)$. We know that (4.29) has a complete set of real eigenvectors corresponding to fixed (σ,ω) and various eigenvalues $i\xi$. We want to show the same thing for $M(\omega,i\xi)$, for fixed (ω,ξ) in Γ and various eigenvalues $i\sigma$.

Suppose that (ω,ξ) is in Γ, and let $\sigma_1,\dots \sigma_k$ denote the eigenvalues of $M(\omega,i\xi)$. For each σ_j choose a basis E_j for the eigenspace of $i\sigma_j A + \sum i\omega_j B_j$ corresponding to the eigenvalue $i\xi$. We are allowing for the possiblity that (4.29) might have multiple eigenvalues. The elements of E_j are also eigenvectors of $M(\omega,i\xi)$ corresponding to the eigenvalue $i\sigma_j$. We claim that the union of the E_j is a complete set of vectors. There are clearly enough of these vectors. The fact that they are linearly independent follows from an argument which is essentially the one which shows that eigenvectors corresponding to distinct eigenvalues are linearly independent. This completes the proof.

We make some comments about the behavior of $M(\omega,i\xi)$ when (ω,ξ) lies on the edge of Γ. According to Figures 4.1 and 4.2, this case corresponds to group velocity which is tangent to the boundary, and it also corresponds to the coalescence of different values of $\sigma(\omega,\xi)$. In addition, as (ω,ξ) approaches the edge of Γ, various eigenvectors of $M(\omega,i\xi)$ associated with the two determinations of $i\sigma(\omega,\xi)$ come together, so that $M(\omega,i\xi)$ fails to have a complete set of eigenvectors. In the case of the Euler and shallow water equations, $M(\omega,i\xi)$ acquires a 2×2 nondiagonalizable block. In systems for which Ω corresponds to multiple eigenvalues, the nondiagonalizable block can be larger. This occurs, for example, with Maxwell's equation. (In this case it is not possible to write $M(\omega,s)$, since det $A = 0$, but there is an analogous matrix which is used. See Section 1 of Majda and Osher [14].) The defective behavior of $M(\omega,i\xi)$ causes major difficulties in the theory of well-posedness. (See Section 6)

Proposition 4.1 and the accompanying remarks have obvious extensions to more general cases, e.g., where the characteristic variety has several different cones. In any system the major point of interest is the behavior of $M(\omega,i\xi)$ when the eigenvalues $i\sigma$ coalesce, that is, at tangential incidence. The examples mentioned here contain this principal difficulty. For the sake of definiteness and clarity, we orient many of the discussions in the remainder of the paper to these examples.

4.4. An algebraic labelling of incoming and outgoing modes

In this section we discuss a process by which incoming and outgoing modes are labelled algebraically in the theory of well-posedness. We consider the elementary solutions

$$e^{i\sigma x + i\omega\cdot y + i\xi t} \tag{4.30}$$

of (4.7) which have been discussed extensively in previous sub-sections. We analyze what happens when the dual variables $i\xi$ and $i\sigma$ are perturbed to the complex values $s = \eta + i\xi$ and $\kappa = \rho + i\sigma$, respectively, so that the resulting form

$$e^{\kappa x + i\omega\cdot y + st} \tag{4.31}$$

is also a solution of (4.7). The sign of Re κ, when Re $s > 0$, will indicate whether the mode (4.30) corresponds to group velocity pointing into or out of the spatial domain $x > 0$. This labelling is used in a fundamental manner in the theory of well-posedness. (See Section 5) The relationships between the modes (4.30) and (4.31) will be discussed more extensively in Section 5.3.

The labelling process depends on an assumption that κ and s are invertible analytic functions of each other for η near zero. This analytic dependence is found in the examples which have been discussed earlier. (See (4.17) and (4.21)) However, it fails for points (σ, ω, ξ) on the cones Ω when (ω, ξ) is on the edge of Γ (i.e., tangential incidence). It also fails for points corresponding to the plane in Figure 4.1 when the plane is horizontal. This situation is encountered when the boundary is characteristic.

<u>Proposition 4.2.</u> Suppose that κ and s have the analytic dependence mentioned above, and suppose that (4.30) is perturbed so that $\operatorname{Re} s > 0$. If (4.30) corresponds to group velocity pointing into the spatial domain $x > 0$, then κ is perturbed so that $\operatorname{Re} \kappa < 0$. If the group velocity points out of the domain, then $\operatorname{Re} \kappa > 0$.

<u>Proof.</u> The vector group velocity is $(-\frac{\partial \xi}{\partial \sigma}, -\nabla_{\omega} \xi)$ (see (4.23)). Incoming and outgoing modes thus correspond to $\partial\xi/\partial\sigma$ negative and positive, respectively. The former case is illustrated in Figure 4.3; the solid arrows indicate that perturbations in ξ and σ have opposite signs.

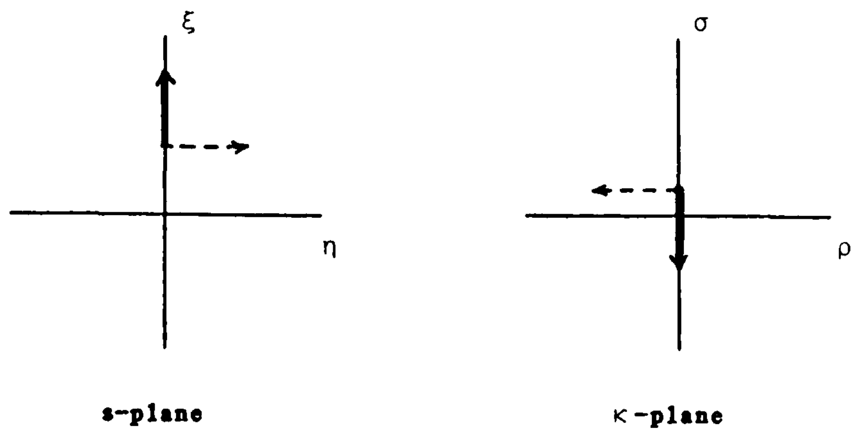


Figure 4.3. The case $\frac{\partial \xi}{\partial \sigma} < 0$.

Because of the analytic dependence, the complex derivative

$\frac{ds}{d\kappa}$ exists and is equal to $\frac{\partial\xi}{\partial\sigma}$. Perturbations in κ and s thus have a ratio which is independent of direction. In the case of Figure 4.3 these perturbations are negatives of each other, so that $\mathrm{Re}\ \kappa < 0$ when $\mathrm{Re}\ s > 0$. This is illustrated by the dotted arrows. Similarly, if (4.30) corresponds to an outgoing group velocity, then $\frac{\partial\xi}{\partial\sigma} > 0$. In this case $\frac{ds}{d\kappa} > 0$, and $\mathrm{Re}\ s > 0$ implies $\mathrm{Re}\ \kappa > 0$. This completes the proof.

The above proof resembles an argument given by Taylor [26,p 202] in a discussion of reflection of singularities. A similar argument was also given by Trefethen in a study of the relationship between group velocity and the stability of boundary conditions for finite difference approximations (e.g., [27], [29]).

We should note that there exist modes (4.31) which cannot be obtained by perturbing (4.30). When $\mathrm{Re}\ s \to 0$ these approach the form $\exp(\kappa x + i\omega' y + i\xi t)$, where $\mathrm{Re}\ \kappa \neq 0$. Such modes are found for (ω, ξ) outside Γ. (See Section 5.3)

5. Interpretations of the Uniform Kreiss Condition

In this section we discuss some effects of the ''Uniform Kreiss Condition'' (U.K.C.). This is the condition (3.20) which was introduced in Section 3.3 and which is sufficient to assure the well-posedness of the IBVP (3.1),(3.3),(3.4). In this section we show that the U.K.C. can be regarded as a solvability condition; it enables one to solve for certain ''incoming'' dependent variables in terms of ''outgoing'' variables and boundary data. The U.K.C. is used for this purpose in proofs of well-posedness. The above interpretation is an analogue to the situation in one space dimension.

The present discussion is based on the use of Fourier transforms and Laplace transforms. The conclusions reached here are therefore limited to systems which have constant coefficients. Analogous conclusions may be expected for the high-frequency portions of solutions to variable-coefficient problems, since the coefficients appear nearly constant to waves whose frequencies are sufficiently high. This principle is contained in the theory of propagation of singularities (e.g., Nirenberg [17], Taylor [26]) and in discussions of ''slowly varying'' wavetrains in the applied literature (e.g., Whitham [31]). However, the variable-coefficient case is technically more complicated, and it may be particularly difficult near tangential incidence. (See Section 5.3)

In any case, the properties of the high-frequency portions of the solution are of major interest; estimates involving L^2 norms and Sobolev norms (e.g., (3.5)) play a major role in the theory, and these norms are governed by behavior at high frequencies.

In Section 5.1 we give an outline of some effects of the U.K.C. This discussion is based on the system (4.2)(a) of ordinary differential equations which was obtained through the use of Fourier and Laplace transforms. The structure of the solutions of this system is discussed in Section 5.2.

In Section 5.3 we discuss in detail the nature of the ''incoming'' and ''outgoing'' modes and how these modes are affected by the U.K.C. The case of tangential incidence is included in this discussion. The development in this sub-section is given in terms of the shallow water equations and the two-dimensional Euler equations.

In Section 5.4 we discuss some weak forms of well-posedness which can be encountered if the U.K.C. is not satisfied completely.

5.1. General remarks

In this sub-section we outline some effects of the U.K.C. The discussion is based on the transformed problem (4.2),

$$(5.1) \quad \begin{aligned} &(a) \quad \hat{u}_x(x,\omega,s) = M(\omega,s)\hat{u} - A^{-1}\hat{F}(x,\omega,s) \\ &(b) \quad \hat{u}^I(0,\omega,s) = S\hat{u}^{II} + \hat{g}(\omega,s), \end{aligned}$$

which was derived earlier.

In order to study the solutions of this problem we transform M to block form. Let $Q(\omega,s)$ be a matrix such that

$$(5.2) \qquad Q^{-1}MQ = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

The transformation is chosen so that the eigenvalues κ of M_1 and M_2 satisfy $\mathrm{Re}\,\kappa < 0$ and $\mathrm{Re}\,\kappa > 0$, respectively. The dimensions of M_1 and M_2 are thus $\ell \times \ell$ and $(n-\ell) \times (n-\ell)$. (cf. Proposition 3.2)

In the theory of well-posedness a great deal of attention is paid to finding block forms which are smooth functions of (ω,s). We will not worry about this now. In (5.2) the matrices Q, M_1, and M_2 are not determined uniquely; this will be discussed below.

The system (5.1)(a) can be written as

$$Q^{-1}\hat{u}_x = Q^{-1}MQQ^{-1}\hat{u} - Q^{-1}A^{-1}\hat{F},$$

or

$$(5.3) \quad \hat{v}_x(x,\omega,s) = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \hat{v} + \hat{G},$$

where $\hat{v} = Q^{-1}\hat{u}$. For convenience we partition $\hat{v}$ into vectors $\hat{v}^I = (\hat{v}_1,\ldots,\hat{v}_\ell)^T$ and $v^{II} = (\hat{v}_{\ell+1},\ldots\hat{v}_m)^T$.

The solutions $\{\hat{v}_1,\ldots\hat{v}_\ell\}$ and $\{\hat{v}_{\ell+1},\ldots,\hat{v}_n\}$ can be constructed from functions of the form $e^{\kappa x}$, where the values κ are eigenvalues of M_1 and M_2, respectively. (See Section 5.2) The solutions $\hat{v}_j$ of (5.1)(a) thus correspond to solutions of the partial differential equation (3.1) which are made up of modes

$$(5.4) \qquad e^{\kappa x + i\omega\cdot y + st}$$

If $1 \leq j \leq \ell$, then $\operatorname{Re} \kappa < 0$. Otherwise, $\operatorname{Re} \kappa > 0$. The labelling process of Proposition 4.2 suggests that the modes (5.4) may be associated with incoming waves if $\operatorname{Re} \kappa < 0$ and outgoing waves if $\operatorname{Re} \kappa > 0$. We may therefore think of $\hat{v}_1,\ldots\hat{v}_\ell$ as ''incoming'' components of the solution and $\hat{v}_{\ell+1},\ldots\hat{v}_n$ as ''outgoing'' components. This identification is developed in greater detail in Section 5.3. The case of tangential incidence is included in this discussion.

The blocks in (5.2) are not determined uniquely, since one can perform similarity transformations on M_1 and M_2 individually. These would amount to linear changes of variables among $\{\hat{v}_1,\ldots\hat{v}_\ell\}$ and $\{\hat{v}_{\ell+1},\ldots,\hat{v}_n\}$. Such transformations do not alter the classes of incoming and outgoing solutions.

We now transform the boundary condition (5.1)(b),

$$[I,-S]\hat{u}(0,\omega,s) = \hat{g}(\omega,s).$$

Since $\hat{v} = Q^{-1}\hat{u}$, this can be written as

$$(5.5) \qquad [I,-S]Q\hat{v} = \hat{g} \qquad (\text{for } x = 0)$$

Let $q_1,\dots q_n$ denote the columns of Q, and let

$$Q^I = [q_1,\dots q_\ell]$$
$$Q^{II} = [q_{\ell+1},\dots q_n]$$

The boundary condition (5.5) is thus

$$[I,-S][Q^I,Q^{II}]\hat{v} = \hat{g},$$

or

$$[I,-S]Q^I\hat{v}^I = -[I,-S]Q^{II}\hat{v}^{II} + \hat{g}$$

This will be denoted by

$$(5.6) \qquad N(\omega,s)\hat{v}^I = P(\omega,s)\hat{v}^{II} + \hat{g} \quad (\text{for } x = 0)$$

It is now possible to see that the U.K.C. can be regarded as a solvability condition. First suppose that $q_1,\dots q_\ell$ are orthonormal, so that $N(\omega,s)$ is a matrix $\bar{N}(\omega,s)$ of the form (3.19). In this case the U.K.C. says explicitly that $|\det N(\omega,s)| \geq \delta$ for $\operatorname{Re} s > 0$, $\omega \in \mathbb{R}^m$. The linear system (5.6) can thus be solved for $\hat{v}^I$, i.e., we can solve for the ''incoming'' variables $\hat{v}_1,\dots\hat{v}_\ell$ in terms of the ''outgoing'' variables $\hat{v}_{\ell+1},\dots\hat{v}_n$ and the boundary data $\hat{g}$.

The same conclusion can be reached, in at least many cases, if $q_1,\dots q_\ell$ are not orthonormal. Some arguments given in Section 3.3 show that if the $q_j(\omega,s)$ are piecewise continuous in (ω,s) and homogeneous of degree zero, then the U.K.C. implies that $|\det N(\omega,s)|$ is bounded away from zero. The linear system (5.6) is thus solvable. In the following discussions we assume that the vectors $q_j(\omega,s)$ are homogeneous. This is not a major restriction, since any scaling of the variables (ω,s) in (5.2) can be confined to M and the blocks M_1 and M_2. Also note comment (3) in Section 3.3.

The solvability condition contains a uniformity which is of interest. Let $s = \eta + i\xi$, fix $\eta > 0$, and consider the limit $|\omega|^2 + |\xi|^2 \to \infty$.

Because $|\det N(\omega,s)| \geq \delta$ for all (ω,s) with $\text{Re } s > 0$, the matrix $N(\omega,s)$ is uniformly invertible as the frequencies tend to infinity.

The significance of this becomes apparent when the U.K.C. is compared with the requirement (3.18), $\det N(\omega,s) \neq 0$ for $\text{Re } s > 0$, which was shown to be necessary for well-posedness. For reasons of homogeneity, $N(\omega,s) = N(\omega',s')$, where

$$(\omega',s') = \frac{(\omega,s)}{(|\omega|^2 + |s|^2)^{1/2}}$$

(See Figure 5.1) The condition (3.18) requires $\det N(\omega',s') \neq 0$ for $\text{Re } s' > 0$, but it allows the possibility that $\det N(\omega',s')$ may tend to zero as $\text{Re } s' \to 0$.

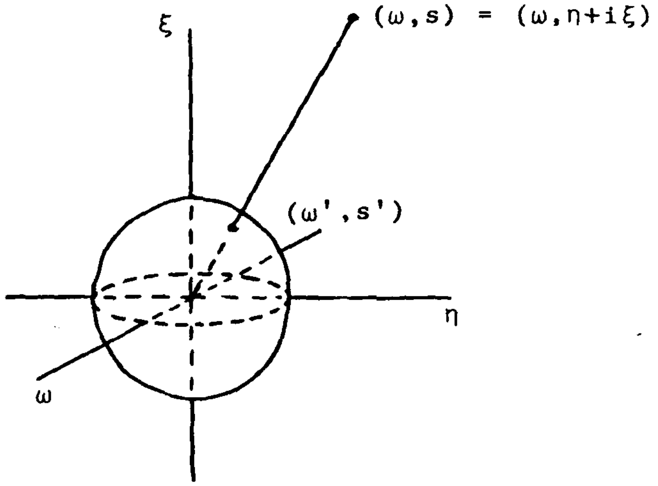


Figure 5.1

That is, for any fixed $\eta > 0$, $N(\omega,s)$ may become more and more singular as $|\omega|^2 + |\xi|^2 \to \infty$. This may have major effects on the solution $\hat{v}^I$ of the system (5.6); unless the right-hand side of (5.6) satisfies special constraints, the ''incoming'' components $\hat{v}^I$ may be large relative to $\hat{v}^{II}$ and $\hat{g}$ for large frequencies. This corresponds to a loss of derivatives at the boundary, or ''weak'' well-posedness. This will be discussed in greater detail in Section 5.4.

In Section 3.1 we assumed that the boundary condition has the special form (3.4),

$$u^I = Su^{II} + g \qquad (\text{for} \quad x = 0)$$

The arguments given in the present section make it possible to justify this assumption. Suppose that we use a more general linear boundary condtion

$$(5.7) \qquad Bu(0,y,t) = h(y,t),$$

where B is a constant matrix. Apply Fourier and Laplace transforms with respect to y and t, respectively, and write the result as

$$B^I\hat{u}^I(0,\omega,s) + B^{II}\hat{u}^{II} = \hat{h}(\omega,s)$$

Here B^I and B^{II} are matrices whose columns are the first ℓ and last $n - \ell$ columns of B, respectively.

Now suppose $\omega = 0$, i.e., consider waves having phase velocities which are normal to the boundary. In this case $M(\omega,s) = sA^{-1}$. Since A is diagonal and has the form (3.2), one can identify $\hat{u}^I$ as the ''incoming'' portion of the solution. If we are to be able to solve for $\hat{u}^I$ in terms of $\hat{u}^{II}$ and $\hat{h}$, then B^I must be an invertible square matrix. Thus B^I is $\ell \times \ell$, and (5.7) can be written as

$$u^I = -(B^I)^{-1}B^{II}u^{II} + (B^I)^{-1}h.$$

This has the form (3.4).

5.2. Solutions of the system (5.1).

In the previous sub-section we used modes (5.4) for which $\operatorname{Re} \kappa > 0$, i.e., which grow exponentially as x increases. This may appear totally unreasonable, since these modes do not have finite energy on the interval

$0 < x < \infty$. In addition, none of the modes (5.4) satisfy the homogeneous initial condition which was used to obtain (4.2)(a) (or (5.1)(a)). In order to make the use of such modes seem a little more legitimate, we therefore derive and discuss some representations of solutions of (5.1). It will become clear that the U.K.C. has a natural interpretation in terms of a two-point boundary value problem associated with (5.1).

We first justify a remark made at the beginning of Section 4. There we assumed that the effects of the initial data f in the IBVP (3.1),(3.3),(3.4) can be absorbed into the forcing term F. This is done in order to facilitate the use of the Laplace transform with respect to t. The preliminary transformation can be accomplished by a procedure used by Rauch [21] in a study of the regularity of solutions of the IBVP.

Suppose that the initial value f vanishes identically in a neighborhood of the space-time corner $x = 0$, $t = 0$; and suppose that the pure initial-value problem for (3.1) is well-posed. There then exists a function w such that

$$w_t = Aw_x + \sum B_j \frac{\partial w}{\partial y_j}$$
$$w(x,y,0) = f(x,y)$$

(We are assuming $C = 0$). Because of the finite propagation speed associated with hyperbolic systems, there is a $\delta > 0$ such that $w(0,y,t) = 0$ if $t \leq \delta$. Choose $h \in C^\infty(\mathbb{R})$ so that $h(0) = 1$ and $h(t) = 0$ for $t \geq \delta$. Let u be a solution of (3.1),(3.3),(3.4); and let $v(x,y,t) = u - h(t)w$. It follows that v satisfies

$$v_t = Av_x + \sum B_j \frac{\partial v}{\partial y_j} + (F - h'(t)w)$$
$$v(x,y,0) = 0$$
$$v^I = Sv^{II} + g, \qquad \text{for } x = 0.$$

This is the form which was used to obtain (4.2). By a re-labelling, we may continue to denote the forcing term by $F(x,y,t)$ and the solution by $u(x,y,t)$. Because of the finite propagation speed, we may assume that $F(x,y,t)$ has compact support with respect to x and y.

The system (5.1)(a) can be solved by studying the block version (5.3). We assume that M_1 and M_2 are themselves block diagonal, where each block is triangular and is associated with a single eigenvalue. If necessary, M_1 and M_2 can be transformed to this form by linear changes of variables among $\hat{v}_1, \dots \hat{v}_\ell$ and $\hat{v}_{\ell+1}, \dots \hat{v}_n$, respectively. Under these assumptions each equation in (5.3) has the form

$$(5.8) \quad \text{or} \quad \begin{array}{ll} \text{(a)} & \dfrac{dw}{dx} = \kappa w + H(x) \\ \text{(b)} & \dfrac{dw}{dx} = \kappa w + \sum c_j w_j + H \end{array}$$

In (5.8)(b) the w_j are solutions of equations which can be solved independently of (5.8)(b). We seek solutions of (5.4) which are in $L^2(0,\infty)$, since we are ultimately considering solutions of the partial differential equation which satisfy energy estimates like (3.5).

We first consider the case $\operatorname{Re} \kappa < 0$. When one multiplies (5.8)(a) by the integrating factor $\exp(-\kappa x)$ and integrates, the result is

$$(5.9) \qquad w(x) = e^{\kappa x} w(0) + \int_0^x e^{\kappa(x-z)} H(z)\,dz.$$

The solution w is in $L^2(0,\infty)$, since $\operatorname{Re} \kappa < 0$ and H has compact support. The first term in (5.9) represents the propagation of the initial data $w(0)$; and the second can be regarded as a superposition of pulses, each of which appears at a point z and is then propagated by the natural frequency in the problem. The more complicated equation (5.8)(b) can be treated in a manner which is similar to the above.

When $\operatorname{Re} \kappa > 0$, the representation (5.9) is not appropriate. In this case the function w in (5.9) would be in $L^2(0,\infty)$ only if the initial value $w(0)$ and the forcing term H satisfy a special relation. Instead, one should impose the condition $w(x) \to 0$ as $x \to \infty$. Under this assumption the solution of (5.8)(a) has the representation

$$(5.10) \qquad w(x) = \int_\infty^x e^{\kappa(x-z)} H(z)\,dz$$

An analogous formula can be found for (5.8)(b).

The system (5.3) (or (5.1)(a)) should thus be associated with a two-point boundary value problem, if the solution is to be in $L^2(0,\infty)$. The components $\hat{v}^I$, corresponding to $\operatorname{Re}\kappa < 0$, should be prescribed at $x = 0$; and the components $\hat{v}^{II}$, corresponding to $\operatorname{Re}\kappa > 0$, should be set to zero at infinity.

We now show that these conclusions make sense in physical terms. The solutions w in (5.9),(5.10) can be factored into the form

$$w(x) = e^{\kappa x}\, c(x), \tag{5.11}$$

for suitable $c(x)$. By inverting Fourier and Laplace transforms, one can see that the solutions of the original partial differential equation are thus made up of functions

$$e^{\kappa x + i\omega' y + st}\, c(x) \tag{5.12}$$

An inspection of the formulas (5.9),(5.10) reveals that these functions are superpositions of pulses which are propagated by modes of the form $\exp(\kappa x + i\omega' y + st)$. According to the labelling process suggested by Propostion 4.2, the cases $\operatorname{Re}\kappa < 0$ and $\operatorname{Re}\kappa > 0$ correspond to ''incoming'' (rightward moving) waves and ''outgoing'' (leftward moving) waves, respectively.

In the two-point boundary value problem the ''leftward moving'' ($\operatorname{Re}\kappa > 0$) components $\hat{v}^{II}$ are set to zero at infinity. This is physically reasonable, since leftward moving waves can arise only when they are stimulated by the inital data or by forcing in the differential equation. The process outlined at the beginning of this sub-section incorporates both of these effects into the ''forcing'' term H in (5.8). In the case $\operatorname{Re}\kappa > 0$ the envelope $c(x)$ in (5.11) and (5.12) has the form

$$c(x) = \int_{\infty}^{x} e^{-\kappa z} H(z)\,dz \tag{5.13}$$

(see (5.10)) This envelope can be nonzero only when x is within the support of H.

''Rightward moving'' (Re $\kappa < 0$) waves may be prescribed at the boundary $x = 0$, and in the interior they may also be stimulated by forcing or by the initial data. Both of these effects are represented in the formula (5.9). In order to properly define the ''incoming'' components at $x = 0$, it is necessary to impose a suitable boundary condition. The U.K.C. describes a class of such conditions; if a boundary condition satisfies this criterion, then it has the effect of expressing the ''incoming'' modes in terms of ''outgoing'' modes and the boundary data (the function g in (5.1)).

The remarks in this sub-section suggest an effect of the labelling process of Proposition 4.2. The cases Re $\kappa < 0$ and Re $\kappa > 0$ force one to make particular choices for the boundary conditions for the system of ordinary differential equations (5.1)(a). The labelling guarantees that these choices are physically reasonable.

5.3 <u>Structure of ''incoming'' and ''outgoing'' modes; behavior at tangential incidence</u>

In Proposition 4.2 and in Section 5.1 and 5.2, we identified modes $\exp(\kappa x + i\omega\cdot y + st)$ as ''incoming'' or ''outgoing'' portions of the solution when Re $\kappa < 0$ and Re $\kappa > 0$, respectively. (Here Re $s > 0$.) In the present sub-section we examine more closely the nature of these modes and the validity of the above labelling. In particular, we discuss the structure of the modes corresponding to tangential group velocity and the effects of the U.K.C. on these modes. We also mention the strictly decaying modes which occur when $(\omega,\xi) \notin \Gamma$. The present discussion is given in terms of the linearized shallow water equations and the linearized two-dimensional Euler equations.

In this sub-section we consider only the solutions of the homogeneous system (4.7), $u_t = Au_x + \sum B_j \frac{\partial u}{\partial y_j}$. In the previous sub-section we noted that these solutions can be used to build up solutions of the more general inhomogeneous system (3.1) (with $C = 0$).

The idea behind Proposition 4.2 is to perturb an oscillatory solution $\exp(i\sigma x + i\omega \cdot y + i\xi t)$ of (4.7) to a solution of the form

$$e^{\kappa x + i\omega \cdot y + st} \tag{5.14}$$

where $\kappa = \rho + i\sigma$ and $s = \eta + i\xi$. The sign of $\mathrm{Re}\,\kappa$, when $\mathrm{Re}\, s > 0$, indicates whether the mode (5.14) is ''incoming'' or ''outgoing''. The Proposition requires that κ and s be invertible analytic functions of each other. As noted earlier, this excludes the case of tangential incidence. We discuss such modes below, but we first consider the analytic case.

The mode (5.14) can be written as

$$e^{\rho x + \eta t}\, e^{i\sigma x + i\omega \cdot y + i\xi t} \tag{5.15}$$

We regard this as an oscillatory mode which is modulated by the amplitude function $A(x,t) = \exp(\rho x + \eta t)$.

<u>Proposition</u> <u>5.1.</u> If the oscillatory part of (5.15) corresponds to group velocity which is not tangent to the boundary, then (5.15) approaches the configuration

$$e^{-\frac{\eta}{v}(x-vt)}\, e^{i\sigma x + i\omega \cdot y + i\xi t} \tag{5.16}$$

as $|\omega| + |\xi| \to \infty$ for fixed $\eta > 0$. Here v is the x-component (i.e., normal component) of the group velocity.

<u>Proof.</u> The exponent in the factor $A(x,t)$ in (5.15) can be written as $\eta(\frac{\rho}{\eta}x + t)$. The ratio $\frac{\rho}{\eta}$ is equal to $\frac{\rho'}{\eta'}$, where $\rho' = \frac{\rho}{|\zeta|}$ and $\eta' = \frac{\eta}{|\zeta|}$; here $\zeta = (\omega,\xi)$ and $|\zeta|^2 = |\omega|^2 + \xi^2$. (This projection is similar to that of Figure 5.1.)

Since $\rho' = 0$ when $\eta' = 0$ in the present case,

$$\frac{\rho'}{\eta'} = \frac{\Delta\rho'}{\Delta\eta'} \approx \frac{d\rho'}{d\eta'}\bigg|_{\eta'=0}$$

The approximation becomes accurate in the limit $\eta' \to 0$, i.e., when $\eta > 0$ is fixed and $|\omega| + |\xi| \to \infty$. Because of the analytic dependence,

$$\frac{d\rho'}{d\eta'}\bigg|_{\eta'=0} = \frac{d\kappa}{ds} = \frac{d\sigma}{d\xi} = -\frac{1}{v}$$

(See (4.22)) The exponent in $A(x,t)$ is thus

$$\eta(\frac{\rho}{\eta}x + t) \approx \eta(-\frac{1}{v}x + t) = -\frac{\eta}{v}(x - vt)$$

The Proposition then follows.

The mode (5.16) is an oscillatory wave which is modulated by an exponential envelope which moves at the normal component of group velocity. If $v > 0$, then (5.16) decays as x increases. As the group velocity approaches tangential incidence, i.e., as $v \to 0$, the mode decays more and more rapidly, so that the effects of the mode are concentrated near the boundary.

If $v < 0$ (i.e., $\mathrm{Re}\,\kappa > 0$), then the *modes* (5.14), (5.15) are ''outgoing'', i.e., leftward moving. According to the remarks in Section 5.2, *modes of this type are stimulated* in the interior by forcing and the effects of initial data. The effects of such stimulation decay rapidly to the left when $v < 0$ and $|v|$ is small (cf. (5.13)).

We now consider the case where the oscillatory part of (5.14), (5.15) has group velocity tangent to the boundary, i.e., $v = 0$.

The discussion in Section 5.1 suggests the effects of the U.K.C. on these tangential modes. In the earlier discussions the modes (5.14) (5.15) have been regarded as ''incoming'' when $\rho = \mathrm{Re}\,\kappa < 0$. According to the remarks in Section 5.1, the U.K.C. says that the boundary condition must give the values of the portion of the solution corresponding to $\mathrm{Re}\,\kappa < 0$. In the present case, this means that the boundary condition must prescribe the behavior of waves moving tangent to the boundary. It is therefore of interest to study the structure of such modes. The main conclusion, Proposition 5.2, will be useful in the discussion of weak well-posedness given in Section 5.4.

In the following discussion of wave propagation near tangential incidence, we consider only the case of constant coefficients. When the coefficients are variable, major complications can arise. For example, the Hamilton-Jacobi equations (4.25) imply that small variations in the coefficients can cause small changes in frequency for a given mode. If a point $(\sigma(\tau),\omega(\tau),\xi(\tau))$ is confined to Ω, and if (ω,ξ) is near the edge of Γ, then the mode can switch quickly from ''incoming'' to ''outgoing'' as the paramenter τ is varied.

Substantial work has been done on the propagation of singularites (i.e., high frequencies) near tangential incidence for various types of equations. The existing theory is quite complicated. See, e.g., Taylor [26] and the references given therein.

We now discuss the form of (5.14),(5.15). Suppose that $(\sigma,\omega,\xi) = (\sigma_0,\omega_0,\xi_0)$ is on Ω and (ω_0,ξ_0) lies on the edge of Γ, so that the oscillatory part of (5.14),(5.15) corresponds to tangential group velocity. (cf. Figures 4.1 and 4.2) The arguments of Propositions 4.2 and 5.1 are not valid in a neighborhood of such a point, since κ and s are not analytic functions of each other. In the case of the shallow water and Euler equations, there is a square root singularity in κ as a function of s. (cf.(4.17)) This is suggested by Figure 5.2; this is a cross-section of the cone Ω corresponding to fixed $\omega = \omega_0$. In particular, the real parts of κ and s satisfy a relation

$$\rho \approx R\sqrt{\eta}$$

for small $\eta > 0$. Here R is a constant.

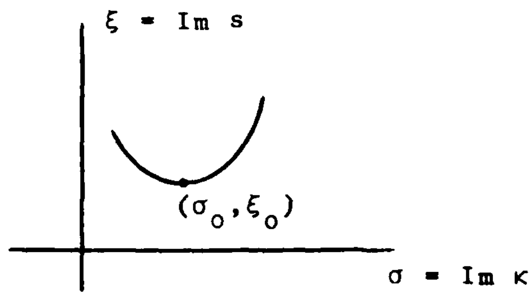


Figure 5.2. Cross-section of Ω for fixed $\omega = \omega_0$.

<u>Proposition 5.2.</u> Suppose that the oscillatory part of (5.15) corresponds to tangential group velocity (i.e., $v = 0$), and suppose that the singularity between κ and s has the form described above. The envelope $A(x,t)$ in (5.15) is then approximately equal to

$$\text{(5.17)} \qquad \exp\left[R\sqrt{\eta|\zeta|}\left(x + \frac{1}{R}\sqrt{\frac{\eta}{|\zeta|}}\,t\right)\right]$$

when the ratio $\frac{\eta}{|\zeta|}$ is small.

<u>Proof.</u> As in Proposition 5.1, the envelope $A(x,t)$ has the form

$$\exp\left[\eta\left(\frac{\rho'}{\eta'}\,x + t\right)\right]$$

In the present case $\rho' \approx R\sqrt{\eta'}$ when η' is small. Since $\eta' = \frac{\eta}{|\zeta|}$, $A(x,t)$ can be written as

$$\begin{aligned} A(x,t) &= \exp\left[\eta\left(\frac{R}{\sqrt{\eta'}}\,x + t\right)\right] \\ &= \exp\left[R\sqrt{\eta|\zeta|}\,x + \eta t\right] \end{aligned}$$

This is equivalent to (5.17). This completes the Proposition.

Now suppose $R < 0$, i.e., $\mathrm{Re}\,\kappa < 0$. In this case the mode (5.15),

$$A(x,t)\ \exp\ (i\sigma x + i\omega' y + i\xi t),$$

decays rapidly away from the boundary. From (5.17) it is apparent that the rate of decay increases without bound as $|\zeta| \to \infty$ with $\eta > 0$ fixed. This situation contrasts with the earlier case $v \neq 0$, where the limiting configuration (5.16) is reached.

The behavior in the present case corresponds to results from the theory of propagation of singularities. In the latter case it is known that as the frequencies tend to infinity, the corresponding portions of the solution follow the bicharacteristic curves (see (4.25)) better and better. In the present case the directions of propagation lie along the boundary.

Up to now we have considered modes (5.14), $\exp\ (\kappa x + i\omega' y + st)$ which are obtained by *perturbing purely oscillatory* modes $\exp(i\sigma x + i\omega\cdot y + i\xi t)$. Such modes correspond to $(\omega,\xi) \in \Gamma$ or points (σ,ω,ξ) on the plane in Figure 4.1. However, there exist modes (5.14) which do not fit this description. These correspond to (ω,ξ) in the exterior of Γ, and for $\text{Re}\ s = 0$ they have the form

$$e^{\kappa x + i\omega' y + i\xi t} \tag{5.18}$$

One may object to labelling such modes as ''incoming'' or ''outgoing'' according to whether $\text{Re}\ \kappa < 0$ or $\text{Re}\ \kappa > 0$, since they do not correspond to the oscillatory propagating waves *mentioned* above. *However*, the U.K.C. still requires that the boundary conditions prescribe the behavior of such modes when $\text{Re}\ \kappa < 0$.

5.4 Weak well-posedness

In this sub-section we discuss some weak forms of well-posedness which can be encountered when the U.K.C. is not quite satisfied.

In Section 5.1 we considered the boundary condition (5.6),

$$N(\omega,s)\hat{v}^{I} = P(\omega,s)\hat{v}^{II} + \hat{g}$$

The U.K.C. (3.20) implies that this system is uniformly solvable for $\hat{v}^I$ as $|\omega| + |\xi| \to \infty$ with $\eta > 0$ fixed (cf. Figure (5.1)). At the end of Section 5.1 we remarked that the weaker necesssary condition (3.18), $\det N(\omega,s) \neq 0$ for $\text{Re } s > 0$, allows the possibility that $\det N(\omega,s)$ may tend to zero as $\text{Re } s \to 0$. In cases where this occurs, the left side of (5.6) becomes singular as $|\omega| + |\xi| \to \infty$ with $\eta > 0$ fixed, so that $\hat{v}^I$ could be large relative to $\hat{v}^{II}$ and $\hat{g}$ at large frequencies. The incoming components would thus be less smooth than g and the other components, i.e., we ''lose derivatives'' at the boundary. In energy estimates like (3.5), derivatives of data would appear on the right side.

Whether this phenomenon actually occurs depends on the structure of the right side of (5.6). For example, it would not occur if $\hat{g} = 0$ and if $P(\omega,s)\hat{v}^{II}$ stays within the range of $N(\omega,s)$ in some suitable sense. An appropriate characterization is that the ratios in Cramer's rule stay bounded as $\text{Re } s \to 0$. However, the loss of derivatives must occur if $P(\omega,s)\hat{v}^{II}$ moves out of the range of $N(\omega,s)$ or if arbitrary g are considered. The degree of the derivatives which are lost depends on the order of the pole in Cramer's rule. An example will be discussed at the end of this sub-section. (See Proposition 5.3)

The above phenomena have a physical interpretation. Suppose $g = 0$, so that the boundary condition (5.6) is a reflection condition, i.e., the ''incoming'' modes $\hat{v}^I$ are reflections of the ''outgoing'' components $\hat{v}^{II}$. The nature of the reflection is governed by the structure of the boundary condition. When derivatives are lost, the amplitudes of the reflected and incident waves have ratios which tend to infinity as $|\omega| + |\xi| \to \infty$. That is, the reflection coefficients tend to infinity. In the other case mentioned above the reflection coefficients remain finite.

The loss of derivatives can cause particular difficulties when the spatial region is a bounded domain rather than a half-space. In this case waves can reflect back and forth between various portions of the boundary, so that more and more derivatives are lost as time progresses.

Trefethen [27], [29] has studied phenomena analogous to the above in connection with the stability of finite difference approximations. He emphasizes the distinction between the cases of finite and infinite reflection coefficients, and he describes how these affect the practical nature of difference methods. In particular, he obtains growth rate estimates for some mild forms of instability.

The case of weak well-posedness is sometimes associated with the term ''generalized eigenvalue''. Recall the eigenvalue problem (3.12),

$$
\text{(5.19)} \qquad
\begin{aligned}
&\text{(a)}\quad s\phi(x) = A\frac{d\phi}{dx} + \Big(\sum_j i\omega_j B_j\Big)\phi \quad ; \; x > 0 \\
&\text{(b)}\quad \phi^{I}(0) = S\phi^{II}(0) \\
&\text{(c)}\quad \phi \in L^2(0,\infty)
\end{aligned}
$$

If $\det N(\omega,s) = 0$ for some (ω,s) with $\operatorname{Re} s = 0$, then a solution ϕ of (5.19)(a)(b) exists. However, it may fail to satisfy the boundary condition at infinity, $\phi \in L^2(0,\infty)$, since the eigenvalues of $M(\omega,s)$ can be purely imaginary when $\operatorname{Re} s = 0$. (See Section 4.3) If this is the case, then the value s in (5.19)(a) is said to be a ''generalized eigenvalue.'' The solution ϕ does happen to be in $L^2(0,\infty)$ if $(\omega,\xi) \in \Gamma$ and if ϕ is associated with the eigenvalue κ of $M(\omega,s)$ for which $\operatorname{Re}\kappa < 0$. (cf.(5.18))

We now describe some examples of weak well-posedness. We mainly consider some examples which were discussed by Kreiss [11] from a point of view which is different from the one expressed above. He studied the system

$$
\text{(5.20)} \qquad u_t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} u_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_y, \qquad u = \begin{pmatrix} u^{I} \\ u^{II} \end{pmatrix} \in \mathbb{R}^2
$$

for $x > 0$, $y \in \mathbb{R}$, $t > 0$, with boundary conditions

$$
\text{(5.21)} \qquad u^{I}(0,y,t) = \alpha u^{II} + g(y,t).
$$

Here α is a complex constant.

A short calculation shows that the eigenvalues κ of $M(\omega,s)$ satisfy

$$
\text{(5.22)} \qquad \kappa^2 = s^2 + \omega^2
$$

For each determination of κ, the corresponding eigenvector is $(\frac{i\omega}{s+\kappa}, 1)^T$. Let κ_0 denote the value of κ which has positive real part, so that $-\kappa_0$ corresponds to the ''incoming'' mode. It then follows that

$$(5.23) \qquad N(\omega,s) = (1, -\alpha)\begin{pmatrix} \frac{i\omega}{s-\kappa_0} \\ 1 \end{pmatrix}$$

(See the derivation of (5.6).) If $N(\omega,s) = 0$, then

$$(5.24) \qquad s = i\left(\frac{\alpha^2+1}{2\alpha}\right)\omega$$

Kreiss discusses three cases where $N(\omega,s) = 0$ for $\mathrm{Re}\ s = 0$:

(1) $\alpha = \pm 1$

(2) $|\alpha| > 1$, α real

(3) $|\alpha| = 1$, α not real

He constructs solutions via Laplace and Fourier transforms under the assumption that the initial condition is $u(x,y,0) = 0$. For case (1) he obtains the estimate

$$(5.25) \qquad \begin{aligned} &\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-2\eta t}\,|u(x,y,t)|^2\,dt\,dx\,dy \\ &\leq \frac{c}{\eta}\int_{-\infty}^{\infty}\int_0^{\infty} e^{-2\eta t}\left(|g|^2 + \left|\frac{\partial g}{\partial t}\right|^2 + \left|\frac{\partial g}{\partial y}\right|^2\right) dt\,dy, \end{aligned}$$

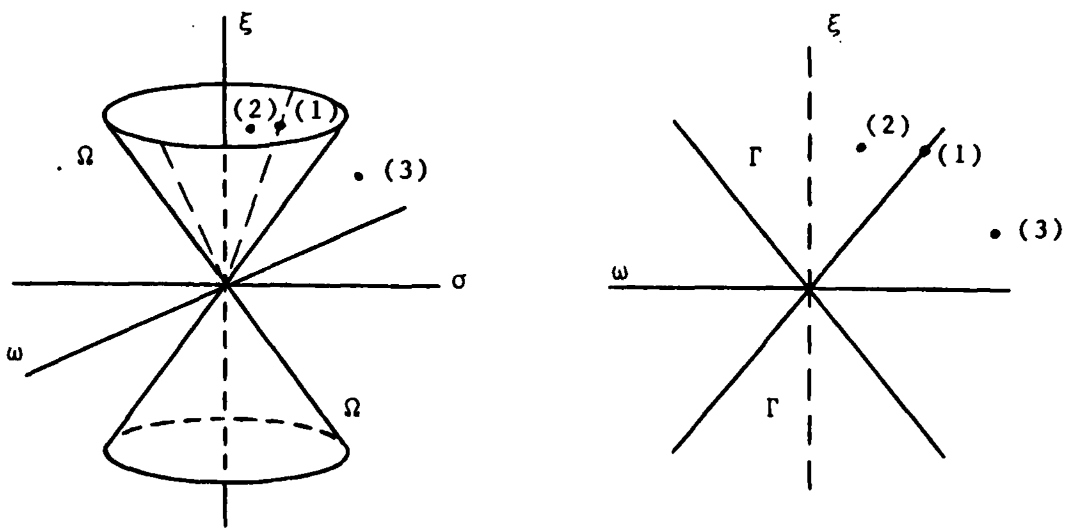
where c is a constant. He also obtains an interior estimate for u in terms of g alone; in this case the spatial domain is given by $x \geq \delta > 0$, $y \in \mathbb{R}$. This estimate is possible because of the rapid decay in x of the modes which cause trouble. In case (2) an estimate like (5.25) is obtained, but is is not possible to obtain a stronger interior estimate. In case (3) the situation is the same as for case (1).

In cases (1) and (2) the solutions ϕ of (5.19)(a)(b) are not in $L^2(0,\infty)$. Kreiss refers to the corresponding values of s as ''generalized eigenvalues of the first kind'' and ''second kind'', respectively. Case (3) corresponds to genuine eigenvalues s.

We now interpret the above results in terms of ideas which have been developed in the present paper. A short calculation shows that the characteristic variety of (5.20) is defined by $\xi^2 = \sigma^2 + \omega^2$, i.e., its graph is like the one in Figure 4.1(a), except that the plane is not present.

In case (1), $N(\omega,s) = 0$ when $s = \pm i\omega$, i.e., $i\xi = \pm i\omega$. (See (5.24) and Figure 5.3) This means that $\kappa = 0$ and that (ω,ξ) lies on the edge of Γ, so that the modes in question correspond to tangential group velocity. The breakdown of the U.K.C. at such points means that the boundary condition does not exert good control over tangential waves. However, the effects of such modes are confined mainly to a neighborhood of the boundary. It is therefore reasonable to expect that a weak estimate like (5.25) would be obtained for $x \geq 0$ but that a stronger estimate should be possible for $x \geq \delta > 0$.

The structure of the tangential modes was described in Proposition 5.2. For the case $\mathrm{Re}\,\kappa < 0$ (''incoming'') the modes decay rapidly away from the boundary, and the rate of decay tends to infinity as $|\omega| + |\xi| \to \infty$. Kreiss observed this kind of behavior in the present example.



(a) (b) Cross-section

Figure 5.3. Locations of points where $N(\omega,s) = 0$ in cases (1), (2), and (3).

We now consider case (2), where $|\alpha| > 1$ with α real. Here $N(\omega, s) = 0$ at points $(\omega, s) = (\omega, i\xi)$ for which $|\xi| > |\omega|$. (See (5.24) and Figure 5.3) In this case (ω, ξ) lies in the interior of Γ, so that the U.K.C. breaks down for frequencies corresponding to non-tangential group velocity. Because the modes in question can influence the interior, one should not expect to recover a stronger estimate by restricting attention to the proper subdomain $x \geq \delta > 0$. The structure of the non-tangential modes was described in Proposition 5.1; as $|\omega| + |\xi| \to \infty$ with $\eta > 0$ fixed, these modes do not decay more and more rapidly, but instead approach the limiting configuration (5.16).

In case (3) we have $|\alpha| = 1$ with α not real. Here $|\xi| < |\omega|$, so that (ω, ξ) lies outside Γ. The corresponding modes are the strictly decaying modes mentioned at the end of Section 5.2. (See (5.18)) For these modes, $\mathrm{Re}\,\kappa \neq 0$ even when $\mathrm{Re}\, s = 0$. Arguments similar to those used in Propositions 5.1 and 5.2 show that when $\mathrm{Re}\,\kappa < 0$, the rate of decay increases without bound as $|\omega| + |\xi| \to \infty$ with $\eta > 0$ fixed. We omit the details. (Also see Kreiss [11].) Thus, as in case (1), a weak estimate like (5.25) is obtained for $x \geq 0$, and a stronger estimate is possible for $x \geq \delta > 0$.

We now examine whether the boundary condition (5.21) for the system (5.20) leads to bounded or unbounded reflection coefficients. For this example the transformed boundary condition (5.6) has the form

$$(5.26) \qquad (1, -\alpha)\begin{pmatrix} \dfrac{i\omega}{s - \kappa_0} \\ 1 \end{pmatrix} \hat{v}^{I} = -(1, -\alpha)\begin{pmatrix} \dfrac{i\omega}{s + \kappa_0} \\ 1 \end{pmatrix} \hat{v}^{II} + \hat{g}$$

After a division and some simplification, (5.26) can be written as

$$\hat{v}^{I} = R(\omega, s)\hat{v}^{II} + \frac{\hat{g}}{N(\omega, s)},$$

where

$$(5.27) \qquad R(\omega, s) = -\frac{i\omega(s - \kappa_0) + \alpha\omega^2}{i\omega(s + \kappa_0) + \alpha\omega^2},$$

and $N(\omega,s)$ is given in (5.23). In order to obtain (5.27) we used the relation (5.22), $\kappa^2 = s^2 + \omega^2$.

$R(\omega,s)$ is the reflection coefficient which relates the ''incoming'' and ''outgoing'' components $\hat{v}^{I}$ and $\hat{v}^{II}$. This coefficient does not play a role in the example as discussed by Kreiss in [11]; in his formulation no outgoing waves can be present, since $u(x,y,0) = 0$ and there is no forcing term.

<u>Proposition 5.3.</u> Let (ω,ξ) be a point such that $N(\omega,i\xi) = 0$, and let $s = \eta + i\xi$. (ω and ξ are fixed.) In case (1), $|R(\omega,s)|$ remains bounded as $\mathrm{Re}\ s \to 0$. In cases (2) and (3), $|R(\omega,s)|$ tends to infinity.

<u>Proof.</u> As $\mathrm{Re}\ s \to 0$, the denominator in $R(\omega,s)$ tends to zero, since $N(\omega,s) \to 0$. The only way that $R(\omega,s)$ can remain bounded is for the numerator also to tend to zero. Thus $-i\omega\kappa_0 = i\omega\kappa_0$ at the point in question. Since $\omega \neq 0$ in cases (1),(2),(3) (cf. (5.25)), we have $\kappa_0 = 0$. The relation $\kappa^2 = s^2 + \omega^2$ ((5.22)) implies $s = \pm i\omega$, which corresponds to case (1). Thus case (1) is the only circumstance in which $|R(\omega,s)|$ could remain bounded.

In this case, $\alpha = \pm 1$. For $s = \eta + i\xi = \eta \pm i\omega$, $R(\omega,s)$ simplifies to

$$R(\omega,s) = -\frac{\eta - \kappa_0}{\eta + \kappa_0}$$

According to remarks made prior to Proposition 5.2, $\kappa_0 \approx c\sqrt{\eta}$ where c is a constant. Thus $R(\omega,s) \to 1$ as $\mathrm{Re}\ s \to 0$. This completes the proof.

The first part of the above proof illustrates the comment made earlier about $P(\omega,s)\ \hat{v}^{II}$ remaining within the range of $N(\omega,s)$.

We conclude by mentioning an example of weak well-posedness studied by Majda and Osher [14, p. 628]. In this case the system of equations is

Maxwell's equations. (See (4.20)) For a class of boundary conditions which includes the perfect conductor boundary condition, they find that the U.K.C. fails at frequencies corresponding to tangential incidence. In their energy estimate which is analogous to (3.5), they have derivatives of the solution and of g on the left and right sides, respectively. The derivatives are taken with respect to y and t.

6. Remarks on proofs of well-posedness.

We now describe some aspects of the process by which the U.K.C. (3.20) is shown to imply that the IBVP (3.1),(3.3),(3.4) is well-posed. This discussion is not self-contained; our purpose is to point to some features of the existing literature and relate them to ideas developed in the present paper.

The main step in the proofs of well-posedness is to produce an *a priori* ''energy estimate'' like (3.5). Such an estimate immediately yields uniqueness of solutions and continuous dependence of solutions on the prescribed data f, F, and g. Existence of solutions is shown via methods of functional analysis. (See, e.g., Section 3 of Majda and Osher [14])

The energy estimate is derived with the aid of a ''symmetrizer''. This is a pseudo-differential operator which has properties specified in, e.g., Kreiss [10, p. 291] and Majda and Osher [14, p. 639]. Once the symmetrizer is constructed the energy estimate is obtained readily. (e.g., [10, p. 281], [14, p. 639])

For problems having constant coefficients, the symmetrizer may be regarded as a smoothly varying matrix function $R(\omega, s)$ which acts on the transformed problem (4.2) discussed earlier. (Here $\omega \in \mathbb{R}^m$, $\mathrm{Re}\ s \geq 0$) In the case of variable coefficients, one uses a corresponding pseudo-differential operator. For simplicity, we use notation appropriate for the former case.

We now make some comments about the construction of $R(\omega, s)$. In comments (3) - (5) we consider systems whose characteristic variety has a structure suggested by Figure 4.1. We will denote by (ω', s') the scaled variables

$$(\omega', s') = (\omega', \eta' + i\xi') = \frac{(\omega, s)}{(|\omega|^2 + |s|^2)^{1/2}}$$

(See Figure 5.1).

(1) One first transforms $M(\omega, s)$ to simple block forms by making appropriate changes of dependent variable, as in (5.2), (5.3). (See later

comments.) In this case the transformations must be smooth functions of (ω,s), so that the calculus of pseudo-differential operators can be used. One then constructs a symmetrizer for the simplified system, and at the end the effects of the preliminary transformations are incorporated into the final symmetrizer. (Note equation (4.5) in Majda and Osher [14] and the equation after (4.8) in Kreiss [10]) The reduction to block forms receives considerable attention in the theory (See Lemma 2.4 in [10], Assumption 1.9 in [14], and Appendix B of Majda [13]).

For different (ω,s) one may need different block forms. The construction is therefore done locally, i.e., one constructs $R(\omega,s)$ in conical neighborhoods in the (ω,s) space and then patches things together via a partition of unity. (See, e.g., equation (4.6) in [14])

(2) The construction is straightforward when $\eta' \geq \eta_0'$, for any fixed $\eta_0' > 0$. (See Section 4 of [10].) In this case the process resembles a method for constructing Liapunov functions for studying the stability of nonlinear autonomous systems of ordinary differential equations (e.g., John [8]). There thus remains only the situation where η' is in a neighborhood of zero.

(3) If (ω,ξ) is in the interior of Γ, and η' is near zero, then $M(\omega,s)$ can be diagonalized (in many cases). One can check that the construction is very straightforward in such a situation.

When $\eta' = 0$, the diagonalizability is given by Proposition 4.1 of the present paper. We thus consider $\eta' \neq 0$. For strictly hyperbolic systems the eigenspaces E_j in Proposition 4.1 are all one-dimensional, so that $M(\omega,i\xi)$ has distinct eigenvalues. Locally, one then has a complete set of eigenvectors which depend analytically on parameters. (See Lemma 2.4 in [10]) Some more general cases are included in the ''block structure'' assumption (Assumption 1.9) of Majda and Osher [14].

(4) For (ω,ξ) outside Γ, $M(\omega,i\xi)$ has eigenvalues κ for which $\mathrm{Re}\,\kappa \neq 0$. (See Section 4.2 of the present paper.) These eigenvalues can

be isolated into blocks and handled as in comment (2) above. The eigenvalues of $M(\omega,s)$ which are purely imaginary when $\mathrm{Re}\ s = 0$ can be treated as in comment (3).

(5) The main difficulty in the construction of $R(\omega,s)$ occurs in neighborhoods of points where (ω,ξ) lies on the edge of Γ and $\eta = 0$. These points correspond to tangential group velocity. We noted in Section 4.3 that $M(\omega,i\xi)$ is defective in this case.

As before, $M(\omega,s)$ is transformed smoothly to block form. The defective block (or blocks) is handled in a complicated manner which is described in Section 4 of Kreiss [10]. The construction is also surveyed in Appendix B of Majda [13]. In the case of the shallow water and Euler equations there is a single defective block of dimension 2×2. One may wish to follow Kreiss' construction for the 2×2 case.

(6) During the course of the construction it is necessary to relate the symmetrizer to the boundary conditions. This is the point in the proof where the U.K.C. is used; this assumption enables one to solve for ''incoming'' ($\mathrm{Re}\ \kappa < 0$) components in terms of ''outgoing'' ($\mathrm{Re}\ \kappa > 0$) components and the boundary data g. (cf. Section 4 of Kreiss [10].)

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We discuss and interpret a theory developed by Kreiss and others for studying the suitability of boundary conditions for linear hyperbolic systems of partial differential equations. The existing theory is extremely technical.

(cont.)

ABSTRACT (cont.)

The present discussion is based on the characteristic variety of the system. The concept of characteristic variety leads to

(1) a physical interpretation of the theory in terms of wave propagation, and

(2) a physical and geometrical method for visualizing the algebraic structure of the system. The great complexity of the theory is caused by certain aspects of this structure.

We also point out connections between the above work and a corresponding theory regarding the stability of finite difference approximations.

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